Learning package

Mathematics\1

For

First year students

By

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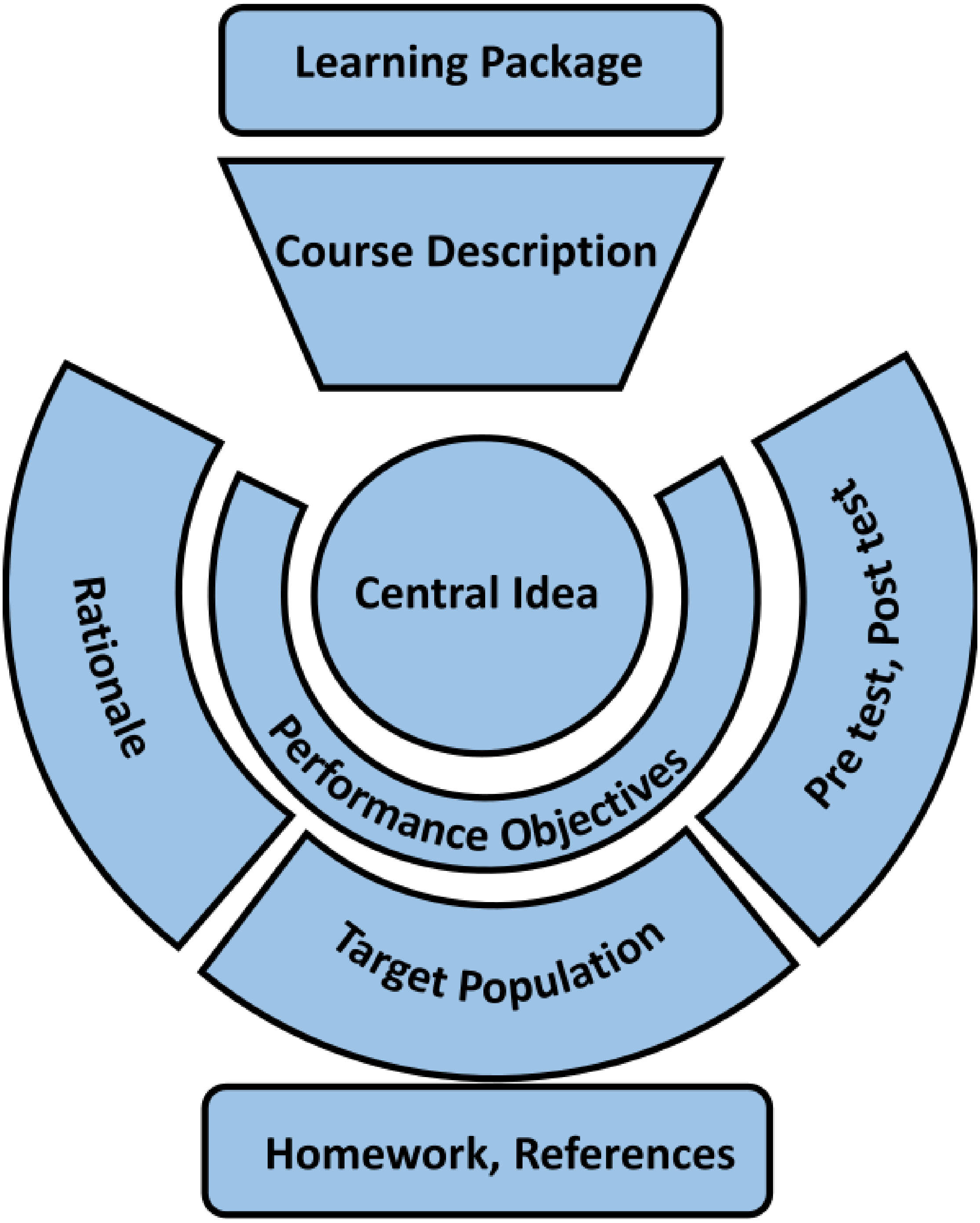
Assistant Lecteror
Dep. Of Surveying techniques
2025


Learning Package
Course Description
Central Idea
Rationale
Performance Objectives
Pre test, Post test
Target Population
Homework, References

Course Description Form

1. Course Name:	
Mathematics\1	
2. Course Code:	
3. Semester / Year:	
2nd semester / first year	
4. Description Preparation Date:	
20/6/2025	
5. Available Attendance Forms:	
Attendance only	
6. Number of Credit Hours (Total) / Number of Units (Total)	
30 hours per course (2 theoretical over 15 weeks) / 2 units	
7. Course administrator's name (mention all, if more than one name)	
Name: Fatima Abdulrazzaq Mohammed	
8. Course Objectives	
Course Objectives Developing the student's skill in employing the principles of mathematics in vario engineering applications and developing their skills to benefit from them in oth engineering lessons.	
9. Teaching and Learning Strategies	
Strategy	1. Cognitive strategies. 2. Active learning strategies. 3. Cooperative learning strategies. 4. Discussion strategy

10. Course Structure

Week	Hours	Required Learning Outcomes	Unit or subject name	Learning method	Evaluation method
1st week	2 hours (theoretical)	1- The student learns about the uses mathematics inengineering applications.	Review in solving equations	Theoretical lectures	Written examsand discussion

2nd week	2 hours (theoretical	2-Developing intellectual, logical a analytical ski to benefit fro them in vario aspects engineering studies.	Matrices typ addition a subtraction matrices.	Theoretical lectures	Written exams and discussion
3rd week	2 hours (theoretical	3-Developing intellectual, logical a analytical ski to benefit fro them in vario aspects engineering studies.	Transpose mat Inverse mat Inverse matrix.	Theoretical lectures	Written exams and discussion
4th week	2 hours (theoretical	4-Developing intellectual, logical a analytical ski to benefit fro them in vario aspects engineering studies.	Binary and terna determinants.	Theoretical lectures	Written exams and discussion

5th week	2 hours (theoretical		Solving simultaneous equations usi determinants.	Theoretical lectures	Written exams and discussion
6th week	2 hours (theoretical		Equation of straight line, tw lines perpendicul two lines parallel	Theoretical lectures	Written exams and discussion
7th week	2 hours (theoretical		Trigonometry Some importa laws trigonometric ratios.	Theoretical lectures	Written exams and discussion
8th week	2 hours (theoretical		Solving a trian Some importa rules for solving triangle.	Theoretical lectures	Written exams and discussion
9th week	2 hours (theoretical		Circular sect circular segme finding area a diameter.	Theoretical lectures	Written exams and discussion
10th week	2 hours (theoretical		Derivative, polynomial functions, impli functions.	Theoretical lectures	Written exams and discussion
11th week	2 hours (theoretical		Derivative trigonometric functions.	Theoretical lectures	Written exams and discussion

12th week	2 hours (theoretical		Derivative applications, finding t equation of t tangent.	Theoretical lectures	Written exams and discussion
13th week	2 hours (theoretical		Integration, integration algebraic functio	Theoretical lectures	Written exams and discussion
14th week	2 hours (theoretical		Integration trigonometric functions.	Theoretical lectures	Written exams and discussion
15th week	2 hours (theoretical		Definite integration, applications definite integrati	Theoretical lectures	Written exams and discussion

11. Learning and Teaching Resources

Required textbooks (curricular books, if an	
Main references (sources)	Thomas' Calculus – G., B., Thomas, M., D., Weir, J. Hass
Recommended books and references (scientific journals, reports…)	Reviewing many scientific journals issued various Iraqi universities, in addition to visits to scientific libraries and the institute's library.
Electronic References, Websites	

Ministry of high Education and Scientific Research
Southern Technical University
Technological institute of Basra
Department of Electronic Techniques

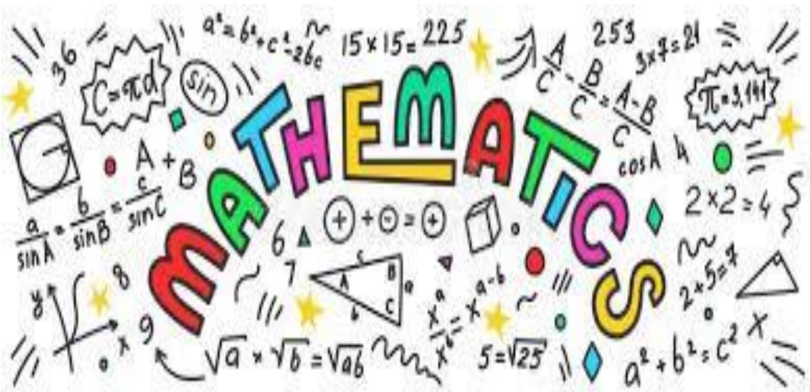
Learning package

In

Mathematics\1

For

Students of First Year



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روابط مهمة

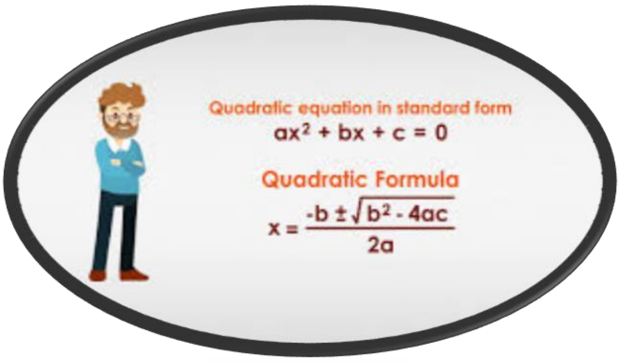
- رابط الكوكل كلاس روم الخاص بقناة (**Mathematics\I**)

رمز دخول : eapffspm

General Objectives

1. To develop students' foundational understanding of core mathematical concepts, including algebra, trigonometry, calculus, and geometry, necessary for engineering applications.
2. To enhance students' analytical and logical reasoning skills by solving mathematical problems that require step-by-step analysis and critical thinking.
3. To equip students with the ability to apply mathematical principles to solve real-world problems in various fields of engineering, particularly surveying and technical disciplines.
4. To introduce students to matrix algebra and its applications, including operations such as addition, subtraction, multiplication, and the computation of inverses and determinants.
5. To familiarize students with differentiation and integration techniques, enabling them to solve problems involving rates of change and area calculations.

Lecture One



Quadratic equation in standard form
ax² + bx + c = 0
Quadratic Formula
x = (-b ± √(b² - 4ac)) / 2a

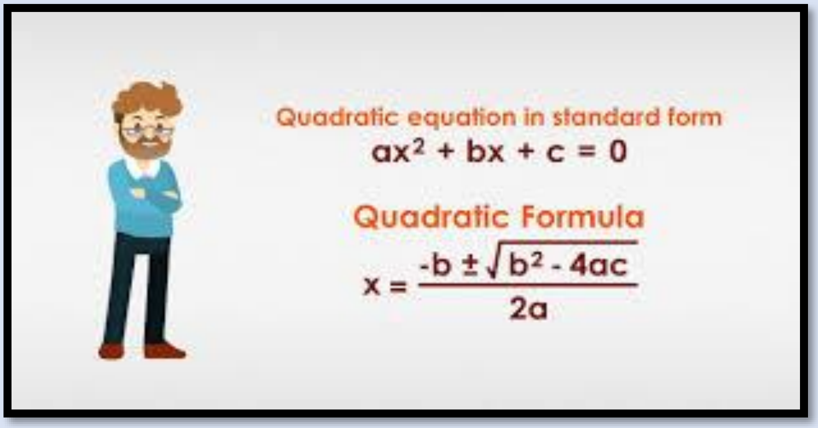
Behavioral objectives

1.Identify different forms of quadratic equations.

2.Solve quadratic equations by factoring, completing the square, and the quadratic formula.

3.Check the correctness of obtained solutions by substitution.

4.Recognize the practical use of quadratic equations in engineering problems.



SOLVING QUADRATIC EQUATIONS

In this brush-up exercise we will review three different ways to solve a quadratic equation.

EXAMPLE 1: Solve: $6x^2 + x - 15 = 0$

SOLUTION

We check to see if we can factor and find that $6x^2 + x - 15 = 0$ in factored form is

$$(2x-3)(3x+5) = 0$$

We now apply the principle of zero products:

$$\begin{array}{r|r} 2x - 3 = 0 & 3x + 5 = 0 \\ 2x = 3 & 3x = -5 \\ x = \dfrac{3}{2} & x = -\dfrac{5}{3} \end{array}$$

We check each solution and see that $x = \frac{3}{2}$ and $x = -\frac{5}{3}$ are indeed solutions for the equation $6x^2 + x - 15 = 0$.

EXAMPLE 2: Solve: $4x^2 + 5x - 6 = 0$

SOLUTION

We can use the quadratic formula to solve this equation. This equation is in standard form, and

$$a = 4 \quad b = 5 \quad c = -6$$

We substitute these values into the quadratic formula and simplify, getting

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(5) \pm \sqrt{(5)^2 - 4(4)(-6)}}{2(4)} = \frac{-(5) \pm \sqrt{25 - (16)(-6)}}{8}$$
$$= \frac{-(5) \pm \sqrt{25 + 96}}{8} = \frac{-5 \pm \sqrt{121}}{8} = \frac{-5 \pm 11}{8}$$

We separate the two solutions and simplify.

$$x = \frac{-5+11}{8} \quad or \quad x = \frac{-5-11}{8}$$

$$= \frac{6}{8} = \frac{3}{4} \qquad\qquad = \frac{-16}{8} = -2$$

So $x = \frac{3}{4}$ and $x = -2$ when $4x^2 + 5x - 6 = 0$. Always check your results.

EXAMPLE 3: Solve by completing the square: $x^2 - x - 6 = 0$

SOLUTION

Step 1: This equation is in standard form. But we want the terms that contain the variable to be on the left and the constant to be on the right. So we add 6 to both sides, obtaining

$$x^2 - x = 6$$

The equation is now in the proper form for completing the square.

Step 2: Because *b* (the coefficient of *x*) is -1, $\frac{b}{2}$ is $\frac{-1}{2}$ and $\left(\frac{b}{2}\right)^2$ is $\left(\frac{-1}{2}\right)^2$

We add this value, $\left(\frac{-1}{2}\right)^2$, to both sides of the equation.

$$x^2 - x + \left(\frac{-1}{2}\right)^2 = 6 + \left(\frac{-1}{2}\right)^2$$

Step 3: We have transformed the left side of the equation into the square of the binomial $\left[x + \left(\frac{-1}{2}\right)\right]^2$. To see that this is so, multiply it out. You get

$$\begin{aligned}\left[x + \left(\frac{-1}{2}\right)\right]^2 &= x^2 + 2(x)\left(\frac{-1}{2}\right) + \left(\frac{-1}{2}\right)^2 \\ &= x^2 + \left(\frac{-2x}{2}\right) + \left(\frac{-1}{2}\right)^2 \\ &= x^2 - x + \left(\frac{-1}{2}\right)^2\end{aligned}$$

We have "completed the square" of the left side.

We write the left side as $\left[x + \left(\frac{-1}{2}\right)\right]^2$ and simplify the right side, obtaining

$$\begin{aligned}\left[x + \left(\frac{-1}{2}\right)\right]^2 &= 6 + \left(\frac{-1}{2}\right)^2 \\ &= 6 + \frac{1}{4} \qquad \text{Squaring} \\ &= \frac{25}{4} \qquad \text{Combining and writing as an improper fraction}\end{aligned}$$

EXERCISES

1. $6n^2 + 30n - 36 = 0$ 2. $y^2 + 11y + 18 = 0$ 3. $x^2 + 13x + 40 = 0$

4. $16r^2 - 24r + 8 = 0$ 5. $4r^2 + 16r - 180 = 0$ 6. $3x^2 + 30x + 27 = 0$

7. $x^2 + 12x + 35 = 0$ 8. $9x^2 - 15x - 6 = 0$ 9. $2x^2 + 22x - 84 = 0$

10. $x^2 - 9x + 18 = 0$ 11. $x^2 - 2x - 15 = 0$ 12. $x^2 + 6x - 7 = 0$

13. $x^2 - 10x - 24 = 0$ 14. $9x^2 - 18x + 8 = 0$ 15. $4x^2 + 36x - 88 = 0$

SOLUTIONS

1. $n = -6$ and $n = 1$ 2. $y = -2$ and $y = -9$ 3. $x = -5$ and $x = -8$

4. $r = \frac{1}{2}$ and $r = 1$ 5. $r = -9$ and $r = 5$ 6. $x = -1$ and $x = -9$

7. $x = -5$ and $x = -7$ 8. $x = -\frac{1}{3}$ and $x = 2$ 9. $x = -14$ and $x = 3$

10. $x = 6$ and $x = 3$ 11. $x = 5$ and $x = -3$ 12. $x = 1$ and $x = -7$

13. $x = 12$ and $x = -2$ 14. $x = 1\frac{1}{3}$ and $x = \frac{2}{3}$ 15. $x = 2$ and $x = -11$

Solving linear equations by substitution

Key points

- Two equations are simultaneous when they are both true at the same time.
- Solving simultaneous linear equations in two unknowns involves finding the value of each unknown which works for both equations.
- Make sure that the coefficient of one of the unknowns is the same in both equations.
- Eliminate this equal unknown by either subtracting or adding the two equations.
- The subsitution method is the method most commonly used for A level. This is because it is the method used to solve linear and quadratic simultaneous equations.

Example 1 Solve the simultaneous equations $y = 2x + 1$ and $5x + 3y = 14$

$5x + 3(2x + 1) = 14$	**1** Substitute $2x + 1$ for y into the second equation.
$5x + 6x + 3 = 14$ $11x + 3 = 14$	**2** Expand the brackets and simplify.
$11x = 11$ So $x = 1$	**3** Work out the value of x.
Using $y = 2x + 1$ $y = 2 \times 1 + 1$ So $y = 3$	**4** To find the value of y, substitute $x = 1$ into one of the original equations.
Check: equation 1: $3 = 2 \times 1 + 1$ YES equation 2: $5 \times 1 + 3 \times 3 = 14$ YES	**5** Substitute the values of x and y into both equations to check your answers.

Example 2 Solve $2x - y = 16$ and $4x + 3y = -3$ simultaneously.

$y = 2x - 16$ $4x + 3(2x - 16) = -3$	**1** Rearrange the first equation. **2** Substitute $2x - 16$ for y into the second equation.
$4x + 6x - 48 = -3$ $10x - 48 = -3$	**3** Expand the brackets and simplify.
$10x = 45$ So $x = 4\frac{1}{2}$	**4** Work out the value of x.
Using $y = 2x - 16$ $y = 2 \times 4\frac{1}{2} - 16$ So $y = -7$	**5** To find the value of y, substitute $x = 4\frac{1}{2}$ into one of the original equations.
Check: equation 1: $2 \times 4\frac{1}{2} - (-7) = 16$ YES equation 2: $4 \times 4\frac{1}{2} + 3 \times (-7) = -3$ YES	**6** Substitute the values of x and y into both equations to check your answers.

Practice questions

Solve these simultaneous equations.

1 $y = x - 4$
$2x + 5y = 43$

2 $y = 2x - 3$
$5x - 3y = 11$

3 $2y = 4x + 5$
$9x + 5y = 22$

4 $2x = y - 2$
$8x - 5y = -11$

5 $3x + 4y = 8$
$2x - y = -13$

6 $3y = 4x - 7$
$2y = 3x - 4$

7 $3x = y - 1$
$2y - 2x = 3$

8 $3x + 2y + 1 = 0$
$4y = 8 - x$

9 Solve the simultaneous equations $3x + 5y - 20 = 0$ and $2(x + y) = \dfrac{3(y - x)}{4}$.

Answers

1 $x = 9, y = 5$

2 $x = -2, y = -7$

3 $x = \frac{1}{2}, y = 3\frac{1}{2}$

4 $x = \frac{1}{2}, y = 3$

5 $x = -4, y = 5$

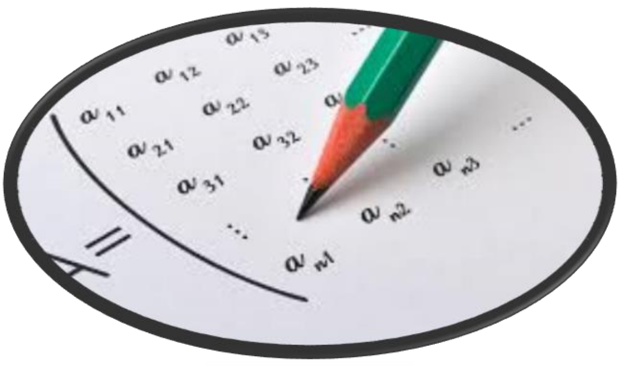
6 $x = -2, y = -5$

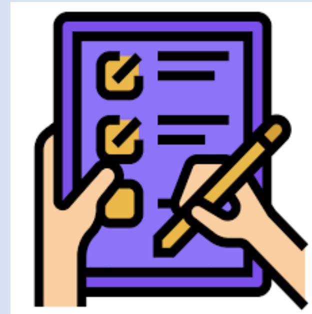
7 $x = \frac{1}{4}, y = 1\frac{3}{4}$

8 $x = -2, y = 2\frac{1}{2}$

9 $x = -2\frac{1}{2}, y = 5\frac{1}{2}$

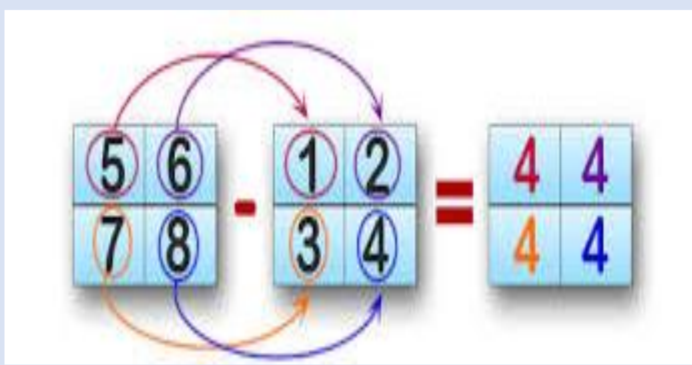
Lecture Two





Behavioral objectives

1. Define matrices and determine their dimensions.
2. Perform addition and subtraction of matrices of the same dimensions.
3. Apply scalar multiplication to matrices.
4. Recognize diagonal and non-diagonal elements of a matrix



Matrices

A matrix is a rectangular grid of numbers arranged into *rows* and *columns*.

An $m \times n$ matrix **A** has *m* rows and *n* columns and is written

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{a_{11}} & \cdots & \boldsymbol{a_{1n}} \\ \boldsymbol{a_{21}} & \dots & \boldsymbol{a_{2n}} \\ \boldsymbol{a_{31}} & \dots & \boldsymbol{a_{3n}} \\ \vdots & \dots & \vdots \\ \boldsymbol{a_{m1}} & \dots & \boldsymbol{a_{mn}} \end{bmatrix}$$

the numbers *m* and *n* are called the dimensions of **A.**

where the element a_{ij} , located in the i^{th} row and the j^{th} column, is a *scalar* quantity; a numerical constant, or a single valued expression.

The *diagonal elements* of a matrix are those elements where the row and column index are the same. For example, the *diagonal elements* of the 3×3 matrix **A** are a_{11}, a_{22}, and a_{33}. The other elements are *non-diagonal* or *off-diagonal* elements.

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

The sum of the diagonal elements of **A** is called the **trace** of the matrix, denoted by **tr(A).**

$$\boldsymbol{tr}(\boldsymbol{A}) = \boldsymbol{a_{11}} + \boldsymbol{a_{22}} + \boldsymbol{a_{33}}$$

A matrix having either a single row ($m = 1$) or a single column ($n = 1$) is defined to be a *vector* because it is often used to define the coordinates of a point in a multi-dimensional space. (In this note the convention has been adopted of representing a vector by a lower case “bold-face” letter such as **x**, and a general matrix by a “bold-face” upper case letter such as **A**.)

A vector having a single row, for example

$$\boldsymbol{x} = [\boldsymbol{x_{11}} \ \ \boldsymbol{x_{12}} \dots \ \ \boldsymbol{x_{1n}}]$$

is defined to be a *row vector*, while a vector having a single column is defined to be a *column vector*

$$\boldsymbol{y} = \begin{bmatrix} \boldsymbol{y_{11}} \\ \boldsymbol{y_{21}} \\ \vdots \\ \boldsymbol{y_{m1}} \end{bmatrix}$$

Special Types of Matrices

1-Square Matrices

Matrices with the same number of *rows* as *columns* (*m*=*n*) are said to be *square* and of order *n* (or *m*).

$$\boldsymbol{A} = \begin{bmatrix} 3 & 2 & 5 \\ i & 0 & -2 \\ -7 & 1 & 8 \end{bmatrix}$$

Here **A** is a square matrix of order 3 (sometimes it is written as $\mathbf{A}_3$).

2- **Diagonal Matrix**

If all *non-diagonal* (*off-diagonal*) elements in a matrix are zero, then the matrix is a diagonal matrix.

$$\boldsymbol{A} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & -9 \end{bmatrix}$$

3- **Identity Matrix**

The *square identity matrix* **I**, is a special matrix where all of its elements equal to zero except those on the main diagonal (where *i* = *j*) which have a value of one:

$$\boldsymbol{I} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

4- **Null Matrix**

The null matrix **0**, which has the value of zero for all of its elements:

$$\mathbf{0} = \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}$$

5- Upper Triangular matrix:

A square matrix said to be an *Upper triangular matrix* if

$$a_{ij} = 0 \; for \; all \; i > j$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$$

6- **Lower Triangular Matrix**:

A square matrix said to be a *Lower triangular matrix* if

$$a_{ij} = 0 \; for \; all \; i < j$$

$$A = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

7- **Symmetric Matrix**:

A square matrix **A** = $(a_{ij})_{nxn}$ said to be a *symmetric matrix* if $a_{ij} = a_{ji}$ for all i and j.

$$A = \begin{bmatrix} 8 & -2 & 6 \\ -2 & 7 & 3 \\ 6 & 3 & 1 \end{bmatrix}$$

8- **Skew- Symmetric Matrix**:

A square matrix **A** = $(a_{ij})_{nxn}$ said to be a *skew-symmetric matrix* if $a_{ij} = -a_{ji}$ for all i and j.

$$A = \begin{bmatrix} 8 & -2 & 6 \\ 2 & 7 & 3 \\ -6 & -3 & 1 \end{bmatrix}$$

Transposition of a Matrix

Consider a matrix **A** with dimensions $m \times n$. The transpose of **A** (denoted $\mathbf{A^T}$) is the $n \times m$ matrix where the columns are formed from the rows of **A**. In other words, $\mathbf{A^T}_{ij} = \mathbf{A}_{ji}$. This “flips” the matrix diagonally

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \\ 5 & 6 \end{bmatrix}, A^T = \begin{bmatrix} 1 & 2 \\ 4 & 3 \\ 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 3 & 6 \end{bmatrix}$$

There are two important observations concerning matrix transposition:

1. For a matrix **A** of any dimension $\mathbf{(A^T)^T = A}$.

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \\ 5 & 6 \end{bmatrix}, A^T = \begin{bmatrix} 1 & 4 & 5 \\ 2 & 3 & 6 \end{bmatrix}, A^{T^T} = \begin{bmatrix} 1 & 2 \\ 4 & 3 \\ 5 & 6 \end{bmatrix}$$

2. For any diagonal matrix **D,** (including the identity matrix **I**), $\mathbf{D}^{\mathbf{T}} = \mathbf{D}$**.**

$$\boldsymbol{D} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & -9 \end{bmatrix}^T = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & -9 \end{bmatrix}$$

3. $(A+B)^T = A^T + B^T$
4. $(A-B)^T = A^T - B^T$

Matrix Algebra

- **Equality of two matrices**
  Two matrices **A** and **B** are said to be equal if
  (i) They are of same dimensions.
  (ii) Their corresponding elements are equal.

- **Addition and Subtraction of Matrices**

It is possible to add two matrices together, but only if they have the same dimensions. If **A** and **B** are both ($m \times n$), then the sum

C = **A + B**

Where **C** is also ($m \times n$) and is defined to have each element the sum of the corresponding elements of **A** and **B**, thus

$$c_{ij} = a_{ij} + b_{ij}$$

Matrix addition is both associative, that is

A + (B + C) = (A + B) + C

and commutative

A + B = B + A

Example:

If **A** and **B** are matrices given by

$$A = \begin{bmatrix} 3 & 2 \\ 4 & -9 \\ -2 & -3 \end{bmatrix}, B = \begin{bmatrix} -3 & 5 \\ 8 & -1 \\ 2 & 8 \end{bmatrix}$$

Find **A + B**

$$A + B = \begin{bmatrix} 3 + (-3) & 2 + 5 \\ 4 + 8 & -9 + (-1) \\ (-2) + 2 & -3 + 8 \end{bmatrix} = \begin{bmatrix} 0 & 7 \\ 12 & -10 \\ 0 & 5 \end{bmatrix}$$

The **subtraction** of two matrices is similarly defined; if **A** and **B** have the same dimensions, then the difference

C = **A** - **B**

Where **C** is also ($m \times n$) and is defined to have each element the difference of the corresponding elements of **A** and **B**, thus

$$c_{ij} = a_{ij} - b_{ij}$$

Example:

If **A** and **B** are matrices given by

$$A = \begin{bmatrix} 3 & 2 \\ 4 & -9 \\ -2 & -3 \end{bmatrix}, B = \begin{bmatrix} -3 & 5 \\ 8 & -1 \\ 2 & 8 \end{bmatrix}$$

Find **A** - **B**

$$A - B = \begin{bmatrix} 3 - (-3) & 2 - 5 \\ 4 - 8 & -9 - (-1) \\ (-2) - 2 & -3 - 8 \end{bmatrix} = \begin{bmatrix} 6 & -3 \\ -4 & -8 \\ -4 & -11 \end{bmatrix}$$

- **Multiplication of a Matrix by a Scalar Quantity**

If **A** is a matrix and k is a scalar quantity, the product

B = k**A**

Where **B** is defined to be the matrix of the same dimensions as **A** whose elements are simply all scaled by the constant k,

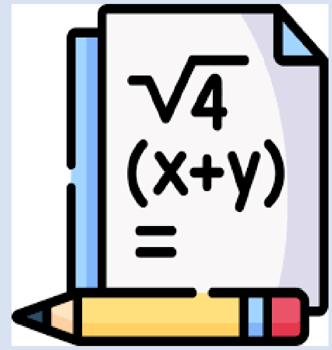
$$b_{ij} = ka_{ij}$$

Example:

$$if\ A = \begin{bmatrix} -3 & 5 \\ 8 & -1 \\ 2 & 8 \end{bmatrix} and\ k = 3, then\ kA = \begin{bmatrix} -9 & 15 \\ 24 & -3 \\ 6 & 24 \end{bmatrix}$$

Lecture Three

Matrices



Behavioral objectives

1. Multiply two matrices when conformable.
2. Use matrix multiplication to solve systems of equations.
3. Transpose a matrix and determine the new dimensions.
4. Apply properties of matrix multiplication in calculations

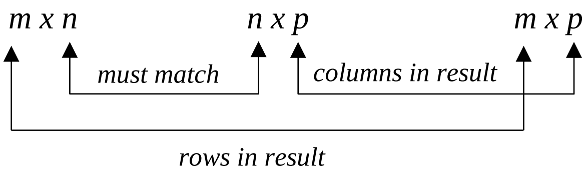
- **Multiplication of two Matrices**

To multiply two matrices their dimensions must meet the condition that allow them to conform, that is the number of columns of the first matrix must be the same as the number of rows as the second one.
Let **A** be a matrix with dimensions $m \times n$, and **B** is another matrix with dimensions $n \times p$, that is the number of columns in **A** is the same as the number of rows in **B**, then the product

C = AB

Where **C** is a matrix with dimensions of *m x p*.

$$A = \begin{bmatrix} ? & ? \\ ? & ? \\ ? & ? \end{bmatrix}, B = \begin{bmatrix} ? & ? & ? \\ ? & ? & ? \end{bmatrix}, C = \begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix}$$



Each element in the resulting matrix **C** will be calculated as

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

The element in position *ij* is the sum of the products of elements in the i^{th} row of the first matrix (**A**) and the corresponding elements in the j^{th} column of the second matrix (**B**).
For example if

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} \text{ then } C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

$$c_{21} = a_{21} \times b_{11} + a_{22} \times b_{21} + a_{23} \times b_{31}$$

Example:
If A and B are matrices given by

$$A = \begin{bmatrix} -1 & 2 & 0 \\ 4 & 1 & -2 \end{bmatrix}, B = \begin{bmatrix} 1 & 3 \\ 0 & -1 \\ 7 & 5 \end{bmatrix}$$

Then find **C=AB**

$$C = \begin{bmatrix} -1 \times 1 + 2 \times 0 + 0 \times 7 & -1 \times 3 + 2 \times -1 + 0 \times 5 \\ 4 \times 1 + 1 \times 0 + -2 \times 7 & 4 \times 3 + 1 \times -1 + -2 \times 5 \end{bmatrix} = \begin{bmatrix} -1 & -5 \\ -10 & 1 \end{bmatrix}$$

General Properties for Matrix Multiplication

1. Matrix multiplication is associative, that is
   A (BC)=(AB) C
2. Matrix multiplication is distributive
   $$(\boldsymbol{A} + \boldsymbol{B})\boldsymbol{C} = \boldsymbol{AC} + \boldsymbol{BC}$$
3. Matrix multiplication is not commutative in general
   $$\mathbf{AB} \neq \mathbf{BA}$$
   in fact, unless the two matrices are square, reversing the order in the product will cause the matrices to be *nonconformal*. The order of the terms in the product is therefore very important. In the product **C = AB, A** is said to *pre-multiply* **B**, while **B** is said to **post-multiply A**.
4. If $\mathbf{A}^2 = \mathbf{I}$**,** then this does not imply that **A = +I or A = -I.**
5. **AB = 0** does not imply that either **A = 0** or that **B = 0.**
6. Even if $\mathbf{A} \neq \mathbf{0}$ and (**AB = AC)** it does not imply that **B =C.**
7. **AI = IA.**
8. $\mathbf{A}^0 = \mathbf{I}$**.**
9. **0A= 0.**

Determinants, Minors and Adjoint

It is a scalar quantity associated with every square matrix. It is denoted as **|A|** (determinant of **A**) and is calculated as

$$\det A = \sum_{k=1}^{n} a_{jk} A_{jk}$$

The summation is carried out on j for any fixed value of k ($1 \leq k \leq n$) or on k for any fixed value of j ($1 \leq j \leq n$). A_{jk} is called the cofactor of the a_{jk} element and is defined as

$$A_{jk} = (-1)^{j+k} M_{jk}$$

Where M_{jk} is called the minor of a_{jk}.

Example

Find the determinant of **A** if you know that

$$A = \begin{bmatrix} 0 & 2 & -1 \\ 4 & 3 & 5 \\ 2 & 0 & -4 \end{bmatrix}$$

Sol.
n = 3

$$\det A = \sum_{k=1}^{3} a_{jk} A_{jk}$$

For j =1

$$\det A = \sum_{k=1}^{3} a_{1k} A_{1k} = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$$

$$\det A = a_{11}(-1)^{1+1}M_{11} + a_{12}(-1)^{1+2}M_{12} + a_{13}(-1)^{1+3}M_{13}$$

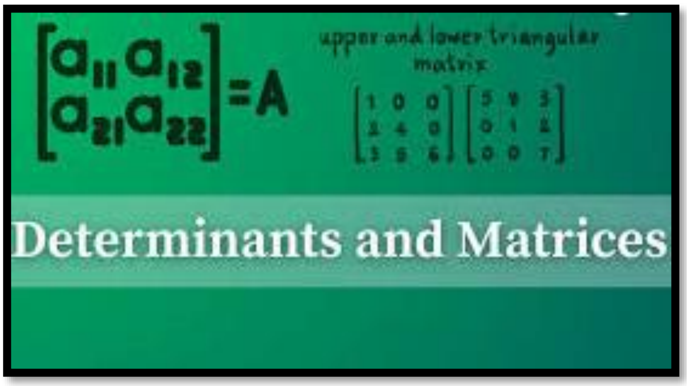
$$\det A = 0\begin{vmatrix} 3 & 5 \\ 0 & -4 \end{vmatrix} - 2\begin{vmatrix} 4 & 5 \\ 2 & -4 \end{vmatrix} + (-1)\begin{vmatrix} 4 & 3 \\ 2 & 0 \end{vmatrix}$$

$$= 0 - 2\big((4 \times -4) - (2 \times 5)\big) + (-1)\big((4 \times 0) - (2 \times 3)\big) = 58$$

Properties of Determinants

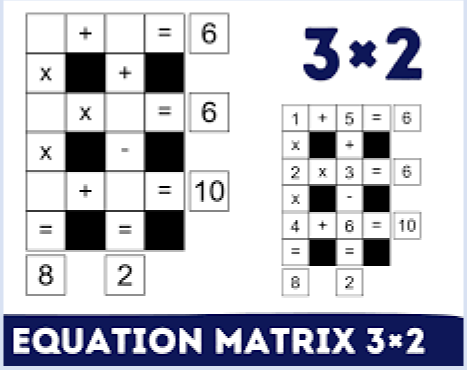
1. ***det I = 1***.
2. If you exchange two rows of a matrix, you reverse the sign of its determinant from positive to negative or from negative to positive.
3. If we multiply one row of a matrix by a scalar k, the determinant is also multiplied by k. e.g.
$$A = \begin{bmatrix} ka & kb \\ c & d \end{bmatrix} = k\begin{vmatrix} a & b \\ c & d \end{vmatrix} = k(ad - cb)$$
4. If A is triangular matrix, then det A is simply the product of the diagonal elements,
$$\boldsymbol{Det\ A = a_{11}\, a_{22}\, a_{33} \ldots a_{nn}}.$$
5. If all elements of any row or column are zero, then det A =0.
6. The det A and its transpose are equal
$$\boldsymbol{\det A = \det A^T}$$
7. In general
$$\mathbf{det}(\boldsymbol{A} + \boldsymbol{B}) \neq \mathbf{det}\,\boldsymbol{A} + \mathbf{det}\,\boldsymbol{B}$$
8. The determinant of a product equals the product of their determinants.
$$\boldsymbol{det(AB) = det\,A\ .\,det\,B}$$
9. If α and β are scalar values then
$$\mathbf{det}(\boldsymbol{\alpha A} + \boldsymbol{\beta}\,\boldsymbol{B}) \neq \boldsymbol{\alpha}\,\mathbf{det}\,\boldsymbol{A} + \boldsymbol{\beta}\,\mathbf{det}\,\boldsymbol{B}$$

Lecture Four



Behavioral objectives

1. Calculate the determinant of 2×2 and 3×3 matrices.
2. Apply cofactor expansion to solve for determinants.
3. Interpret the meaning of a zero determinant.
4. Use determinants in evaluating the solvability of systems of equations.



Inverse of a Matrix

When a system of equations or linear system is defined in a matrix form as

$$\boldsymbol{Ax = c}$$

A is a matrix representing the variables coefficients, **x** is a vector of variables, and **c** is a vector of scalar values.
For example

$$x_1 + x_2 + 2x_3 = 4$$
$$3x_1 + x_2 - 3x_3 = -4$$
$$2x_1 - 3x_2 - 5x_3 = -5$$

The system of equations above can be expressed as

$$\begin{bmatrix} 1 & 1 & 2 \\ 3 & 1 & -3 \\ 2 & -3 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \\ -5 \end{bmatrix}$$

Then to solve the system and find **x** we need to find (**A^{-1}**):

$$\mathbf{x = A^{-1}c}$$
$$\textbf{where } \mathbf{A^{-1} \neq (1 / A)}$$

It's simply a matrix where

$$\mathbf{A^{-1}.A= I \text{ or } A.A^{-1}=I}$$

Inverse of a Matrix using Adjoint Method

$$\boldsymbol{A^{-1}} = \frac{\textbf{adj } \boldsymbol{A}}{\textbf{det} \boldsymbol{A}}$$

Where (**adj A**) is the adjoint matrix of **A** and (**det A**) is a scalar value denotes the determinant of **A**.
The *adjoint* of **A** is defined as the transpose of the *cofactor matrix*

$$\textbf{adj } \boldsymbol{A} = [\boldsymbol{cofactor}]^{\boldsymbol{T}}$$

To find the cofactor elements you need to struck out the minors of the matrix adding their signs (+or -) according to its position or index creating a new matrix called the cofactor matrix.

Cofactor matrix is a matrix of the minors

$$cofactor = (-1)^{i+j} M_{ij}$$

Where M_{ij} is the matrix of minors.

Example

If you know that **A** is a matrix such that

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 0 & 1 & 4 \\ 1 & 5 & -2 \end{bmatrix}, then\ Find\ A^{-1}?$$

Sol.

$$\boldsymbol{A^{-1}} = \frac{\textbf{adj}\,\boldsymbol{A}}{\textbf{det}\,\boldsymbol{A}}$$

To find **det A**

n=3

$$\det A = \sum_{k=1}^{3} a_{jk} A_{jk}$$

For j=1

$$\det A = 3\begin{vmatrix} 1 & 4 \\ 5 & -2 \end{vmatrix} - 2\begin{vmatrix} 0 & 4 \\ 1 & -2 \end{vmatrix} - 1\begin{vmatrix} 0 & 1 \\ 1 & 5 \end{vmatrix}$$

$$= 3[(1\times -2) - (5\times 4)] - 2[(0\times -2) - (1\times 4)] - 1[(0\times 5) - (1\times 1)]$$
$$= 3[-2-20] - 2[-4] - 1[-1] = (3\times -22) + 8 + 1 = -66 + 9 = -57$$

$$cofactor = \begin{bmatrix} \begin{vmatrix} 1 & 4 \\ 5 & -2 \end{vmatrix} & -\begin{vmatrix} 0 & 4 \\ 1 & -2 \end{vmatrix} & \begin{vmatrix} 0 & 1 \\ 1 & 5 \end{vmatrix} \\ -\begin{vmatrix} 2 & -1 \\ 5 & -2 \end{vmatrix} & \begin{vmatrix} 3 & -1 \\ 1 & -2 \end{vmatrix} & -\begin{vmatrix} 3 & 2 \\ 1 & 5 \end{vmatrix} \\ \begin{vmatrix} 2 & -1 \\ 1 & 4 \end{vmatrix} & -\begin{vmatrix} 3 & -1 \\ 0 & 4 \end{vmatrix} & \begin{vmatrix} 3 & 2 \\ 0 & 1 \end{vmatrix} \end{bmatrix}$$

$$cofactor = \begin{bmatrix} -22 & 4 & -1 \\ -1 & -5 & -13 \\ 9 & -12 & 3 \end{bmatrix}$$

$$adj = \begin{bmatrix} -22 & 4 & -1 \\ -1 & -5 & -13 \\ 9 & -12 & 3 \end{bmatrix}^T = \begin{bmatrix} -22 & -1 & 9 \\ 4 & -5 & -12 \\ -1 & -13 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{-1}{57}\begin{bmatrix} -22 & -1 & 9 \\ 4 & -5 & -12 \\ -1 & -13 & 3 \end{bmatrix}$$

For verification find $A.A^{-1} = I$

Properties of A^{-1}

1. If $\boldsymbol{A}$ and $\boldsymbol{B}$ are invertible matrices then
$$(\boldsymbol{AB})^{-1} = \boldsymbol{B}^{-1}.\boldsymbol{A}^{-1}$$
2. If $\boldsymbol{A}$ is invertible then
$$(\boldsymbol{A}^{\boldsymbol{T}})^{-1} = (\boldsymbol{A}^{-1})^{\boldsymbol{T}}$$
$$and\ \mathbf{det}\,\boldsymbol{A}^{-1} = \frac{\mathbf{1}}{\mathbf{det}\,\boldsymbol{A}}$$
3. If $\boldsymbol{A}$ is invertible then
$$(\boldsymbol{A}^{-1})^{-1} = \boldsymbol{A}$$

H.W.

Find A^{-1} for the following matrices

1. $\boldsymbol{A} = \begin{bmatrix} 1 & 3 & 0 \\ -2 & 3 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

2. $\boldsymbol{A} = \begin{bmatrix} 2 & -3 & 0 \\ 5 & -4 & 1 \\ 7 & 2 & 0 \end{bmatrix}$

3. $\boldsymbol{A} = \begin{bmatrix} 4 & 2 & 0 \\ -1 & -2 & -1 \\ 0 & 5 & 3 \end{bmatrix}$

Cramer's Rule

If
$$\boldsymbol{Ax} = \boldsymbol{c}$$
where $\boldsymbol{A}$ is invertible, then each component $\boldsymbol{x_i}$ of $\boldsymbol{x}$ may be computed as the ratio of two determinants, the denominator is **det A**, and the numerator is also the determinant of $\boldsymbol{A}$ matrix but with $\boldsymbol{i^{th}}$ column replaced by $\boldsymbol{c}$.

Example:

Solve the following system of equations using cramer's rule
$$x_1 + 3x_2 = 5$$
$$-2x_1 + 3x_2 + x_3 = 1$$
$$x_2 + x_3 = -2$$

Sol.
First the system of equations are represented as a 3x3 matrix as follows

$$\begin{bmatrix} 1 & 3 & 0 \\ -2 & 3 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ -2 \end{bmatrix}$$

To find x_1 using cramer's rule

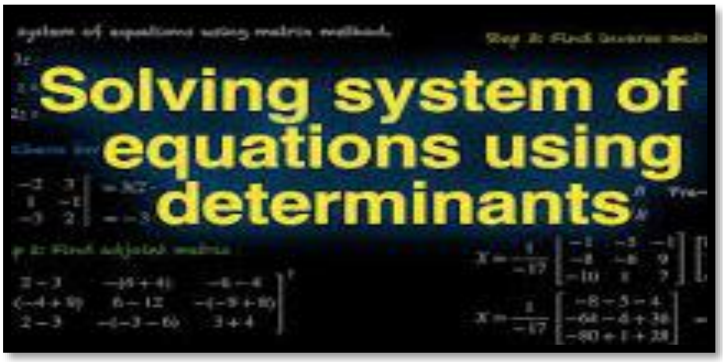
$$x_1 = \frac{\begin{vmatrix} 5 & 3 & 0 \\ 1 & 3 & 1 \\ -2 & 1 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 3 & 0 \\ -2 & 3 & 1 \\ 0 & 1 & 1 \end{vmatrix}} = \frac{1}{8}$$

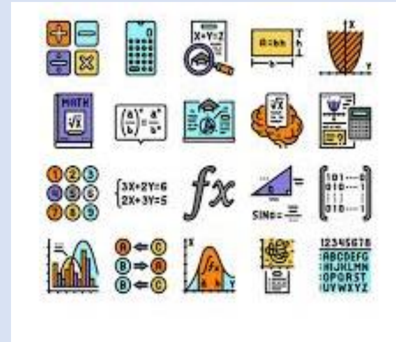
$$x_2 = \frac{\begin{vmatrix} 1 & 5 & 0 \\ -2 & 1 & 1 \\ 0 & -2 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 3 & 0 \\ -2 & 3 & 1 \\ 0 & 1 & 1 \end{vmatrix}} = \frac{13}{8}$$

$$x_3 = \frac{\begin{vmatrix} 1 & 3 & 5 \\ -2 & 3 & 1 \\ 0 & 1 & -2 \end{vmatrix}}{\begin{vmatrix} 1 & 3 & 0 \\ -2 & 3 & 1 \\ 0 & 1 & 1 \end{vmatrix}} = \frac{-29}{8}$$

$$\begin{vmatrix} 1 & 3 & 0 \\ -2 & 3 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 1(3 \times 1) - (1 \times 1) - 3[(-2 \times 1) - 0] + 0 = 8$$

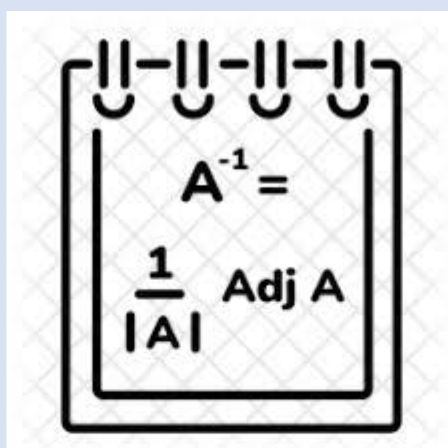
Lecture Five





Behavioral objectives

1. Set up coefficient, variable, and constant matrices from systems of linear equations.
2. Calculate Δ, Δx, Δy (and Δz for 3-variable systems).
3. Solve linear systems using Cramer’s Rule.
4. Verify the correctness of solutions through substitution



Solution to simultaneous linear equation using determinant method

Equations in two unknown

Matrix $= \begin{pmatrix} 2 & 3 \\ 1 & -5 \end{pmatrix}\begin{pmatrix} 12 \\ -7 \end{pmatrix}$

$\Delta = \begin{vmatrix} 2 & 3 \\ 1 & -5 \end{vmatrix} = [2 \times (-5) - 3 \times 1]$

$= -10 - 3$

$= -13$

$\Delta x = \begin{vmatrix} 12 & 3 \\ -7 & -5 \end{vmatrix} = [12 \times (-5) - 3 \times (-7)]$

$= -60 + 21$

$= -39$

$\Delta y = \begin{vmatrix} 2 & 12 \\ 1 & -7 \end{vmatrix} = [2 \times (-7) - 12 \times 1]$

$= -14 - 12$

$= -26$

$X = \frac{\Delta x}{\Delta}$ $\qquad Y = \frac{\Delta y}{\Delta}$

$= \frac{-39}{-13}$ $\qquad = \frac{-26}{-13}$

X= 3 $\qquad Y = 2$

(1) Matrix $= \begin{pmatrix} 2 & -1 & 1 \\ 1 & 3 & 2 \\ 3 & -1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 16 \\ 3 \end{pmatrix}$

$\Delta = \begin{vmatrix} 2 & -1 & 1 \\ 1 & 3 & 2 \\ 3 & -1 & 1 \end{vmatrix} = 2\begin{vmatrix} 3 & 2 \\ -1 & 1 \end{vmatrix} - (-1)\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} + 1\begin{vmatrix} 1 & 3 \\ 3 & -1 \end{vmatrix}$

$= 2[3 - (-2)] + 1(1 - 6) + (-1 - 9)$

$= 2(5) + 1(-5) + (-10)$

$= 10 - 5 - 10$

$= -5$

$$\Delta x = \begin{vmatrix} 0 & -1 & 1 \\ 16 & 3 & 2 \\ 3 & -1 & 1 \end{vmatrix} = 0\begin{vmatrix} 3 & 2 \\ -1 & 1 \end{vmatrix} - \begin{vmatrix} 16 & 2 \\ 3 & 1 \end{vmatrix} + \begin{vmatrix} 16 & 3 \\ 3 & -1 \end{vmatrix}$$

$$= 0[3 - (-2)] + 1(16 - 6) + 1(-16 - 9)$$

$$= 0(5) + 1(10) + (-25)$$

$$= 0 + 10 - 25$$

$$= -15$$

$$\Delta y = \begin{vmatrix} 2 & 0 & 1 \\ 1 & 16 & 2 \\ 3 & 3 & 1 \end{vmatrix} = 2\begin{vmatrix} 16 & 2 \\ 3 & 1 \end{vmatrix} - 0\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 16 \\ 3 & 3 \end{vmatrix}$$

$$= 2(16 - 6) - 0(1 - 6) + (3 - 48)$$

$$= 2(10) - 0(-5) + (-45)$$

$$= 20 + 0 - 45$$

$$= -25$$

$$\Delta z = \begin{vmatrix} 2 & -1 & 0 \\ 1 & 3 & 16 \\ 3 & -1 & 3 \end{vmatrix} = 2\begin{vmatrix} 3 & 16 \\ -1 & 3 \end{vmatrix} - (-1)\begin{vmatrix} 1 & 16 \\ 3 & 3 \end{vmatrix} + 0\begin{vmatrix} 1 & 3 \\ 3 & -1 \end{vmatrix}$$

$$= 2[9 - (-16)] + 1(3 - 48) + 0(-1 - 9)$$

$$= 2(25) + 1(-45) + 0(-10)$$

$$= 50 - 45 + 0$$

$$= 5$$

$X = \frac{\Delta x}{\Delta}$ $\qquad$ $Y = \frac{\Delta y}{\Delta}$ $\qquad$ $Z = \frac{\Delta z}{\Delta}$

$= \frac{-15}{-5}$ $\qquad$ $= \frac{-25}{-5}$ $\qquad$ $= \frac{5}{-5}$

$= 3$ $\qquad$ $= 5$ $\qquad$ $= -1$

Class work

(1) Solve the simultaneous linear equations using determinant method

$$4x - 2y = 9\,; \quad x + y = 3$$

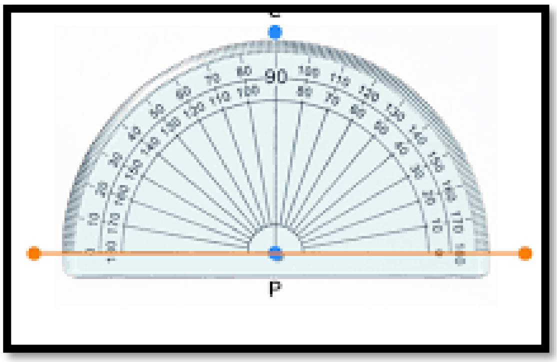
Assignment

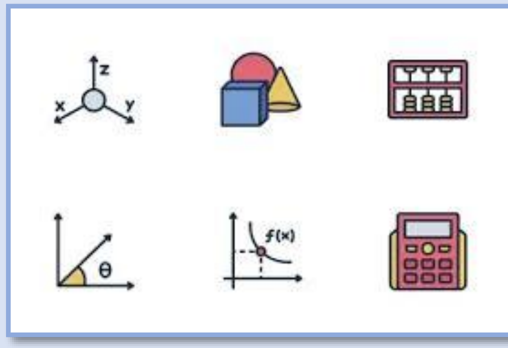
(1) Solve the simultaneous linear equations using determinant method

$3x - 4y + 8z = 26;\ \ 6x + 3y - 5z = 1;\ -2x + y + 3z = 11$

(2) *Find the value of x when* $\begin{vmatrix} 1 & 0 & 2 \\ 3 & x & 5 \\ 2 & 7 & 1 \end{vmatrix} = -5$

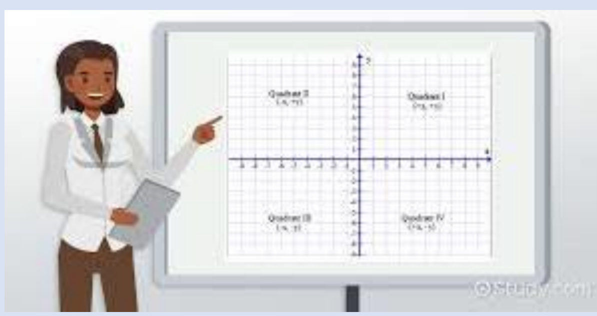
Lecture Six





Behavioral objectives

1. Derive the slope of a line from two points.
2. Find the equation of a line in slope-intercept and point-slope forms.
3. Determine whether two lines are parallel or perpendicular using slope properties.
4. Interpret geometric meanings of slope, intercept, and angle between lines.



Co-ordinate Geometry, The Line

The equations that will be given in the Formulae and Tables book are:

The Line:

Slope: $= \frac{rise}{run}$

Slope: $m = \frac{y_2 - y_1}{x_2 - x_1}$

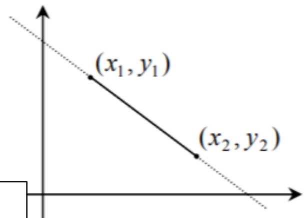
Distance $= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Midpoint $= \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

Use when given slope and a point on the line or two points on the line.

Equation of a line: $y - y_1 = m(x - x_1)$ or $y = mx + c$

Use when given slope and y-intercept



Area of a triangle with one vertex at the origin $= \frac{1}{2}|x_1 y_2 - x_2 y_1|$

Point dividing a line segment in the ratio $a : b$ $= \left(\frac{bx_1 + ax_2}{a + b}, \frac{by_1 + ay_2}{a + b}\right)$

Distance from a point to a line $= \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$

Angles between two line of slopes m_1 and m_2 : $tan\theta = \pm \frac{m_1 - m_2}{1 + m_1 m_2}$

All of the above formulae are given in the Formulae and Tables Booklet. However, there is some other key information that you must be aware of:

Slopes:

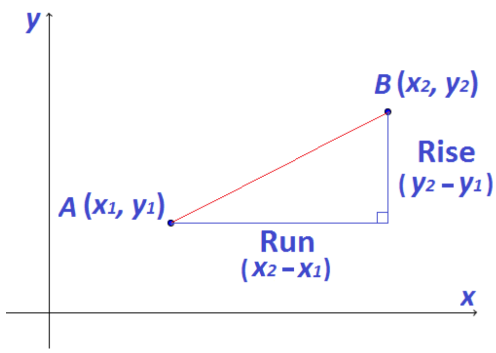
The **slope** or gradient of a line is a value that describes both the direction and the steepness of the line.

The given diagram shows the points $A(x_1, y_1)$ and B (x_2, y_2)

$$\text{Slope} = \frac{\text{Rise}}{\text{Run}}$$

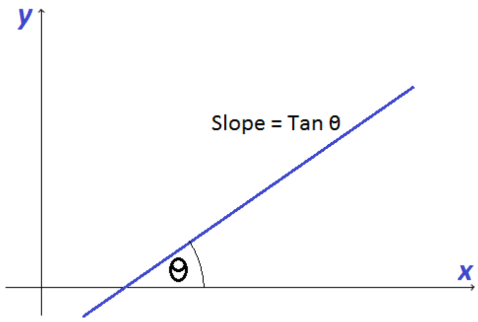
The slope, m, of the line passing through the points (x_1, y_1) and (x_2, y_2) is

$$\text{Slope} = m = \frac{y_2 - y_1}{x_2 - x_1}$$



The slope of a line is also defined as the tangent of the angle between the line and the positive sense of the *x*-axis.

Slope = Tan θ



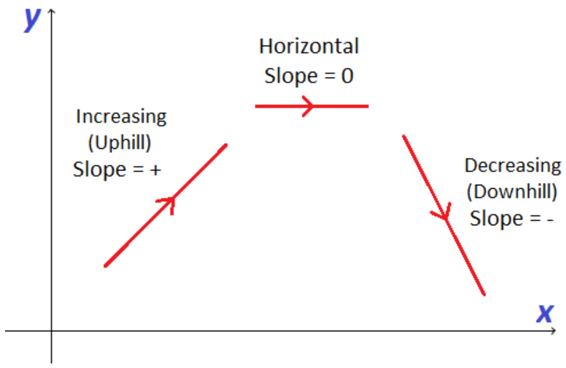
Positive and Negative Slopes

Graphs are read from left to right.

If a line is increasing (going uphill), it has a positive slope.

A horizontal line has a slope of zero.

If a line is decreasing (going downhill), it has a negative slope.



Parallel Lines

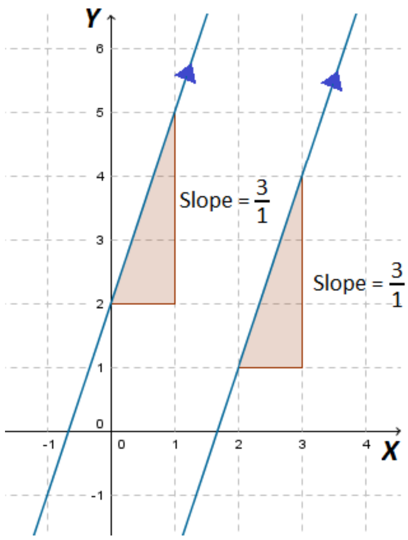
If two lines are parallel if their slopes are equal.

$$\text{If } l_1 || l_2 \text{ then } m_1 = m_2$$

In the given diagram, the two lines are parallel, therefore their slopes are equal:

$$\frac{3}{1} = \frac{3}{1}$$

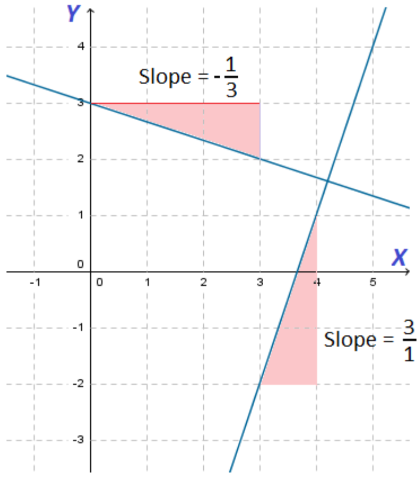
If two lines are parallel, their slopes are equal.



Perpendicular Lines

If two lines are perpendicular, when we multiply their slopes we get -1.

$$\text{If } l_1 \perp l_2 \text{ then } (m_1)(m_2) = -1$$



In the given diagram, the two lines are perpendicular, therefore the product of the slopes is:

$$-\frac{1}{3} \times \frac{3}{1} = -\frac{3}{3} = -1$$

If two lines are perpendicular, the product of their slopes is -1.

1. **Note:** If we know the slope of a line and we need to find the slope of a line perpendicular to it, we turn the given slope upside down and change the sign.

2015 Paper 2 Question 3

(a) The co-ordinates of two points are $A(4, -1)$ and $B(7, t)$.

The line $l_1: 3x - 4y - 12 = 0$ is perpendicular to AB. Find the value of t.

Slope $AB = \frac{t+1}{7-4} = \frac{t+1}{3}$ Slope $l_1 = \frac{3}{4}$

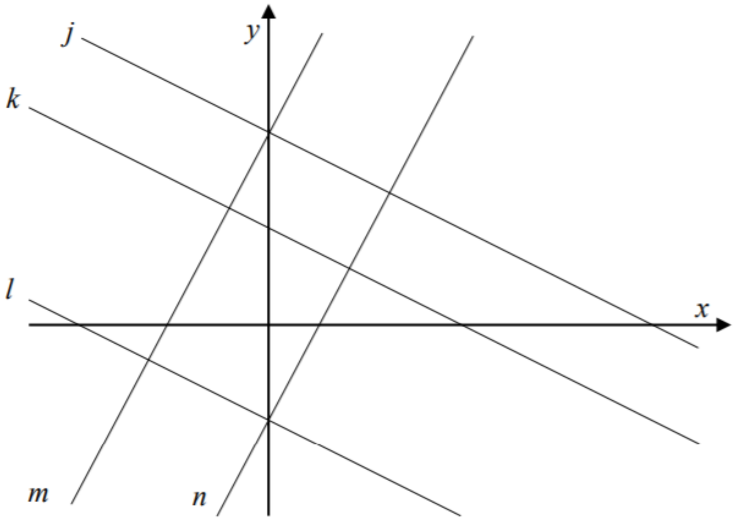
$AB \perp l_1 \Rightarrow \frac{t+1}{3} \times \frac{3}{4} = -1 \Rightarrow t+1 = -4 \Rightarrow t = -5$

Since they are perpendicular, we find each slope and then use m_1 x m_2 = -1

When using the equation $\boldsymbol{y = mx + c}$, m = the slope and c = y-intercept

2011 Paper 2 Question 3

In the co-ordinate diagram shown, the lines j, k, and l are parallel, and so are the lines m and n. The equations of four of the five lines are given in the table below.



Equation	Line
$x+2y=-4$	
$2x-y=-4$	
$x+2y=8$	
$2x-y=2$	

Rearrange each equation so that it's in the form $y = mx + c$

$x+2y=-4 \Rightarrow \quad y=-\frac{1}{2}x-2 \quad \rightarrow l$

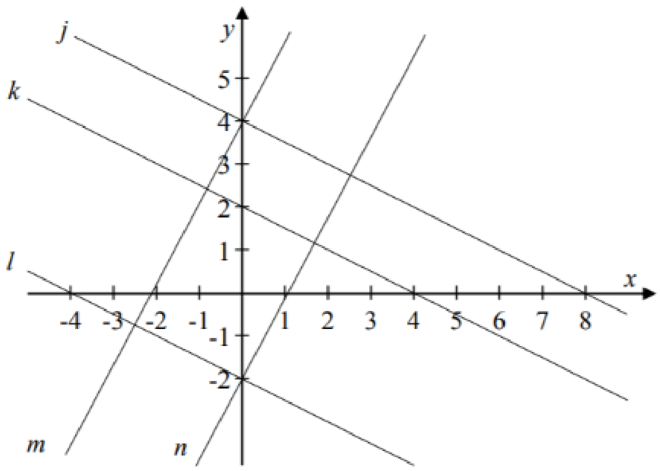
$2x-y=-4 \Rightarrow \quad y=2x+4 \quad \rightarrow m$

$x+2y=8 \Rightarrow \quad y=-\frac{1}{2}x+4 \quad \rightarrow j$

$2x-y=2 \Rightarrow \quad y=2x-2 \quad \rightarrow n$

(a) Complete the table, by matching four of the lines to their equations.

(b) Hence, insert scales on the x-axis and y-axis.



Points on a Line:

Substitute the coordinates of the point in for *x* and *y* in the equation of the line.
If the equation then balances, the point is on the line (LHS = RHS).
If the equation does not balance, the point is not on the line (LHS ≠ RHS).

2018 Paper 2 Question 5

The line $m: 2x + 3y + 1 = 0$ is parallel to the line $n: 2x + 3y - 51 = 0$.

(a) Verify that $A(-2, 1)$ is on m.

(b) Find the coordinates of B, the point on the line n closest to A, as shown below.

(b) Find the coordinates of B, the point on the line n closest to A, as shown below.

Here we must find the equation of the line *AB* and then find where the lines *AB* and *n* intersect.

Slope of *m* or *n* = $\frac{-2}{3}$

To find a slope that is perpendicular to another slope, turn the first slope upside down and change the sign.

Slope of *AB* is $\frac{3}{2}$ and (−2, 1) is on *AB*

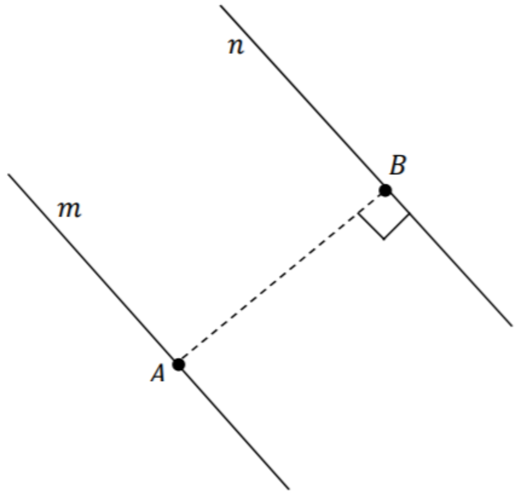
$$y - 1 = \frac{3}{2}(x - (-2))$$

equation of *AB* is $3x - 2y + 8 = 0$

Solve for (x, y) between

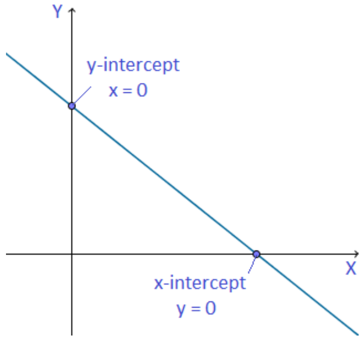
$3x - 2y + 8 = 0$ and $2x + 3y - 51 = 0$

$n \cap AB = (6, 13) = B$



The point of intersection is found using simultaneous equations.

Sketching Lines:



To sketch any line we need any **two points** on the line.

The easiest points to find are the *x* and *y*- intercepts:

To find the point where a line cuts the ***x*-axis we let *y* = 0**

To find the point where a line cuts the ***y*-axis we let *x* = 0**

To find the point of intersection of two lines we use **simultaneous equations**

Equation of a line

To find the equation of a line, we need a point on the line (x_1, y_1) and the slope of the line, m.

The equation of a line can then be found by using the formula:

$$y - y_1 = m(x - x_1)$$

It is good practice to write the completed equation of the line in the form $ax + by + c = 0$, where $a, b, c \in \mathbb{R}$.

2019 Paper 2 Question 2

(a) The line p makes an intercept on the x-axis at $(a, 0)$ and on the y-axis at $(0, b)$, where $a, b \neq 0$.
Show that the equation of p can be written as $\frac{x}{a} + \frac{y}{b} = 1$.

$m = \frac{b-0}{0-a} = \frac{-b}{a}$

$y - y_1 = m(x - x_1)$

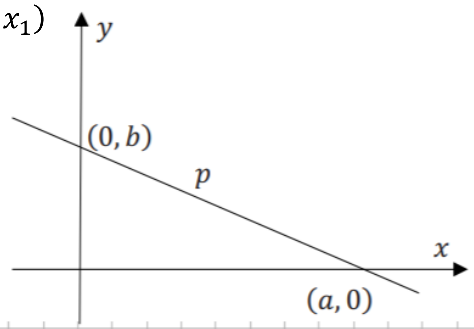
$y - 0 = \frac{-b}{a}(x - a)$

$ay = -bx + ab$

$bx + ay = ab$

Now divide across by ab

$$\frac{x}{a} + \frac{y}{b} = 1$$



(b) The line l has a slope m, and contains the point $A(6, 0)$.

(i) Write the equation of the line l in terms of m.

$$y - y_1 = m(x - x_1)$$

$y - 0 = m(x - 6)$ <u>or</u> $y = m(x - 6)$

Or

$$y = mx - 6m$$

Or

$$y = mx + c$$

$\therefore 0 = 6m + c \Rightarrow c = -6m$

To find the slope when given the equation of the line

Method 1:

When an equation of a line is written in the form $ax + by + c = 0$, with all terms on the left-hand side, then

$$\text{Slope} = -\frac{\text{Number in front of } x}{\text{Number in front of } y}$$

For a line in the form $\boldsymbol{ax + by + c = 0}$

$$\text{Slope} = -\frac{a}{b}$$

Method 2:

When an equation of a line is written in the form $y = mx + c$,
with the y on its own on the left-hand side, then

$$\text{Slope} = m = \text{number in front of the } x$$

For a line in the form $\boldsymbol{y = mx + c}$

$$\text{Slope} = m$$

y-intercept = c

The ***y*-intercept** is the point where the line crosses the y-axis.

Finding the equation of Parallel and Perpendicular lines

To find the equation of a line parallel or perpendicular to a given line:

1) Find the slope of the given line.
2) Find the slope of the required parallel or perpendicular line
3) Use the given point and the new slope to find the equation of the required line

Remember:
* Parallel lines have equal slopes
* The product of the slopes of perpendicular lines equals -1.

A line parallel to $ax + by + c = 0$ will be in the form $ax + by + d = 0$

A line perpendicular to $ax + by + c = 0$ will be in the form $bx - ay + d = 0$

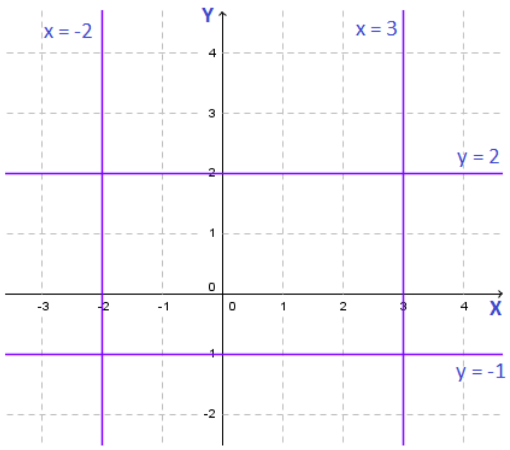
Lines that are parallel to the axes

Some lines are parallel to the x-axis (horizontal) and some lines are parallel to the y-axis.

$x = 0$ is the equation of the y-axis.
$y = 0$ is the equation of the x-axis.

The line $x = a$ is vertical and passes through the value a on the x-axis.

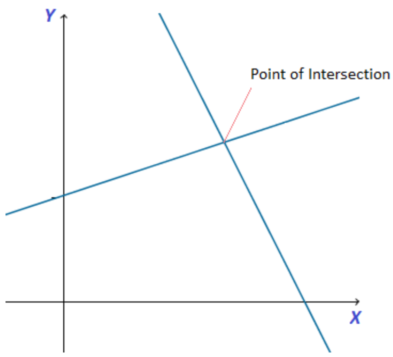
The line $y = b$ is horizontal and passes through the value b on the y-axis.



Point of Intersection of Two Lines

To find the point of intersection of two lines you could graph the lines on the coordinated plane and read off the point of intersection.

However, it can be quicker and possibly more accurate to find the point of intersection using Algebra. To do this, write the equation of each line in the form $ax + by = c$, where $a, b, c \in \mathbb{R}$. Solve these two equations simultaneously, to find a value for x and a value for y. The point of intersection is then (x, y)

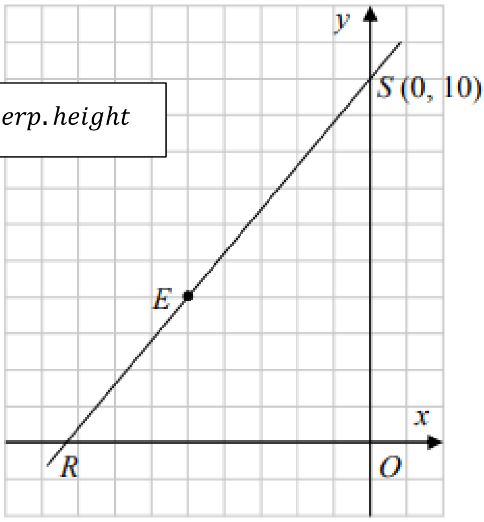


Use your knowledge of solving simultaneous equations from Algebra. Remember that there are two methods for solving simultaneous equations:
Method 1: Elimination method
Method 2: Substitution method

Remember that the symbol ∩ means intersection.

2016 Paper 2 Question 5

The line *RS* cuts the *x*-axis at the point *R* and the *y*-axis at the point $S(0, 10)$, as shown. The area of the triangle *ROS*, where *O* is the origin, is $\frac{125}{3}$.



Area of a Triangle = $\frac{1}{2} \times base \times perp.height$

(a) Find the co-ordinates of *R*.

Area $ROS = \frac{1}{2}|RO|.|OS| = \frac{125}{3}$

$\Rightarrow \frac{1}{2}|RO|(10) = \frac{125}{3}$

$\Rightarrow |RO| = \frac{25}{3}$

$R\left(-\frac{25}{3}, 0\right)$

(b) Show that the point $E(-5, 4)$ is on the line *RS*.

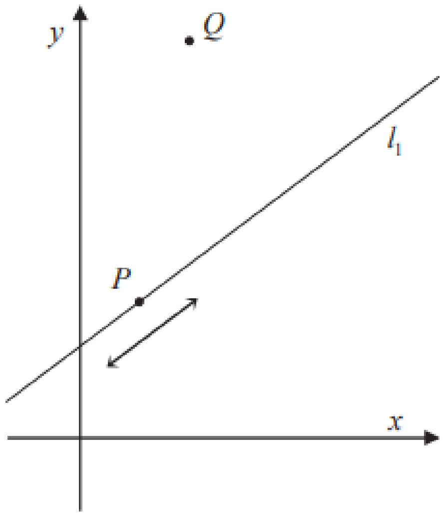
Find the equation of the line first and then check if *E* is on the line.

RS: $y - 10 = \frac{6}{5}(x - 0) \Rightarrow 6x - 5y + 50 = 0$

$6(-5) - 5(4) + 50 = -30 - 20 + 50 = 0 \Rightarrow (-5, 4) \in RS$

2014 Sample Paper 2 Question 2

(a) Show that, for all $k \in \mathbb{R}$, the point $P(4k-2, 3k+1)$ lies on the line $l_1 : 3x - 4y + 10 = 0$.



If $(x,y) = (4k-2, 3k+1)$ then

$$\begin{aligned} 3x - 4y + 10 &= 3(4k-2) - 4(3k+1) + 10 \\ &= 12k - 6 - 12k - 4 + 10 \\ &= 0 \end{aligned}$$

So the equation of l_1 is satisfied. Therefore $(4k-2, 3k+1)$ lies on l_1.

(b) The line l_2 passes through P and is perpendicular to l_1. Find the equation of l_2, in terms of k.

$3x - 4y + 10 = 0$
$-4y = -3x - 10$
$y = \frac{3}{4}x + \frac{5}{2}$
Therefore the slope of l_1 is $\frac{3}{4}$. Therefore l_2 has a slope of $-\frac{4}{3}$ and a point of $(4k-2, 3k+1)$. So it has equation:
$y - (3k+1) = -\frac{4}{3}(x - (4k-2))$
$3y - 3(3k+1) = -4(x - (4k-2))$
Resulting in: $4x + 3y - 25k + 5 = 0$

(c) Find the value of k for which l_2 passes through the point $Q(3, 11)$.

The equation of l_2 is

$$4x + 3y - 25k + 5 = 0.$$

Now $(3,11)$ lies on l_2 if and only if $4(3) + 3(11) - 25k + 5 = 0 \Leftrightarrow 25k = 50 \Leftrightarrow k = 2$. So the $k = 2$ is the required value.

(d) Hence, or otherwise, find the co-ordinates of the foot of the perpendicular from Q to l_1.

When $k = 2$ the equation of l_2 is

$$4x + 3y - 45 = 0.$$

Solving the equations:

$$\begin{aligned} 1.\ 3x - 4y + 10 &= 0 \\ 2.\ 4x + 3y - 45 &= 0 \end{aligned}$$

$$\begin{aligned} 1.\ 12x - 16y + 40 &= 0 \\ 2.\ 12x + 9y - 135 &= 0 \end{aligned}$$

$$-25y + 175 = 0.$$

Therefore $25y = 175$ and $y = \frac{175}{25} = 7$.
Now $3x - 4(7) + 10 = 0 \Leftrightarrow 3x = 4(7) - 10 = 18 \Leftrightarrow x = 6$. So the foot of the perpendicular from Q to l_1 has co-ordinates $(6, 7)$.

Area of a Triangle given its vertices:

Find the area of a triangle with vertices (1, 5), (-3, 1) and (3, -5). $\frac{1}{2}|x_1y_2 - x_2y_1|$

In order to find the area of this triangle we must **translate** one of the points to the origin, (0, 0) and the other points must follow the same translation.

In this case we take 1 from each *x*-value and 5 from each *y*-value.

Let (1, 5) → (0, 0)
(-3, 1) → (-4, -4)
(3, -5) → (2, -10)

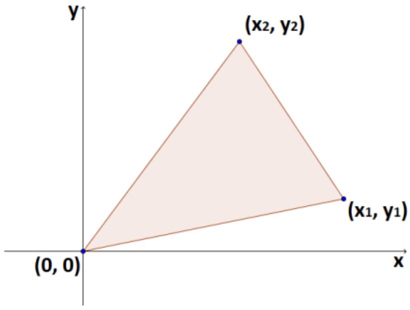
Area of a triangle $= \frac{1}{2}|x_1y_2 - x_2y_1|$

$= \frac{1}{2}|(-4)(-10) - (2)(-4)|$

$= \frac{1}{2}|40 + 8|$

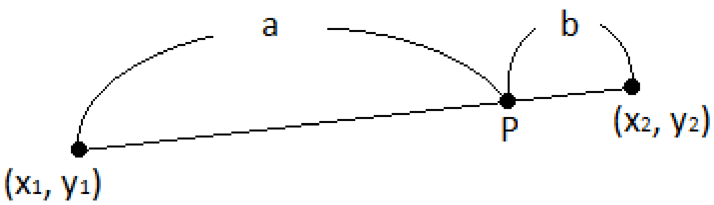
$= \frac{1}{2}|48|$

= 24 square units



Division of a Line segment internally in the ratio of a:b:

In the given diagram, the point P divides the line segment [AB] in the ratio a : b.



The coordinates of P are given by the formula:

$$P = \left(\frac{bx_1 + ax_2}{b + a}, \frac{by_1 + ay_2}{b + a}\right)$$

Ex. 1 Find the coordinates of the point which divides the line segment *A*(-1, 3) and *B*(4, -2) internally in the ratio of *a*:*b* = 2:1.

$\left(\frac{ax_2+bx_1}{a+b}, \frac{ay_2+by_1}{a+b}\right)$

A(-1, 3) and *B*(4, -2)

(x_1, y_1) (x_2, y_2)

$= \left(\frac{ax_2+bx_1}{a+b}, \frac{ay_2+by_1}{a+b}\right)$

$= \left(\frac{(2)(4)+(1)(-1)}{2+1}, \frac{(2)(-2)+(1)(3)}{2+1}\right)$

$= \left(\frac{8-1}{3}, \frac{-4+3}{3}\right)$

$= \left(\frac{7}{3}, \frac{-1}{3}\right)$

Ex. 2

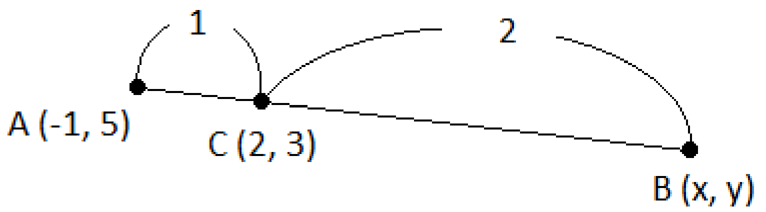
A (-1, 5) and *B* (*x*, *y*) are two points on a plane. The point *C* (2, 3) divides [*AB*] in the ratio 1 : 2.

Find the values of *x* and *y*.

Solution

Method 1 – using the formula

Sketch the line segment:



A (-1, 5) = (x_1, y_1) *B* (*x*, *y*) = (x_2, y_2)

1 = *a* 2 = *b*

$$P = \left(\frac{bx_1 + ax_2}{b+a}, \frac{by_1 + ay_2}{b+a}\right)$$

$$(2, 3) = \left(\frac{(2)(-1) + (1)(x)}{2+1}, \frac{(2)(5) + (1)(y)}{2+1}\right)$$

$$(2, 3) = \left(\frac{-2 + x}{3}, \frac{10 + y}{3}\right)$$

Equate the coordinates:

x coordinates:	*y* coordinates
$2 = \frac{-2+x}{3}$	$3 = \frac{10+y}{3}$
$6 = -2 + x$	$9 = 10 + y$
$8 = x$	$-1 = y$

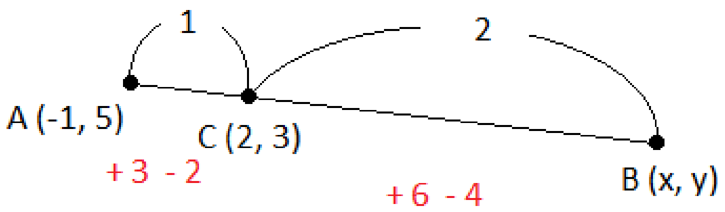
Therefore, *B* (8, -1)

Method 2 – using translations

The translation to go from A to C is: Add 3 to x coordinate and subtract 2 from y coordinate

Using the ratios we can see that the translation from C to B is twice that of the translation from A to C.

Therefore, the translation to go from C to B is: Add 6 to x coordinate and subtract 4 from y coordinate



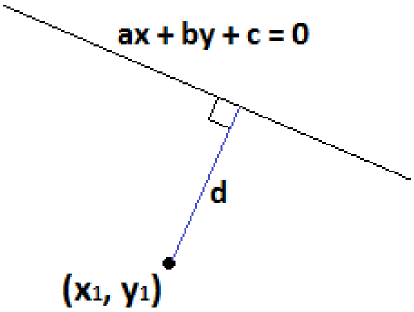
Translations are seen in transformation geometry.

$$C(2, 3) \rightarrow (2 + 6, 3 - 4) \rightarrow B(8, -1)$$

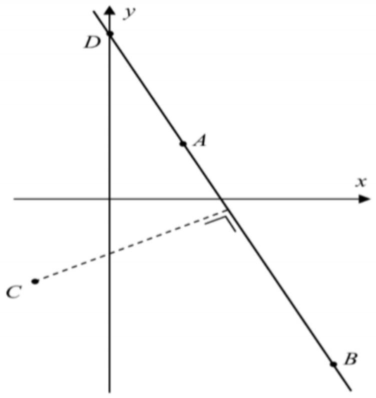
Perpendicular Distance Formula:

The perpendicular distance, d, from the point (x_1, y_1) to the line $ax + by + c = 0$ is given by the formula

$$d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$



2012 Sample Paper



The co-ordinates of three points *A*, *B*, and *C* are: *A*(2, 2), *B*(6, –6), *C*(–2, –3).
(See diagram on facing page.)

(a) Find the equation of *AB*.

Here $(x_1, y_1) = (2,2)$ and $(x_2, y_2) = (6,-6)$. First find the slope of AB

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-6-2}{6-2} = \frac{-8}{4} = -2$$

Now use the equation of a line formula

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 2 &= -2(x-2) \\ y - 2 &= -2x + 4 \\ 2x + y - 6 &= 0 \end{aligned}$$

(c) Find the perpendicular distance from *C* to *AB*.

Using the perpendicular distance formula:

$$\frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

From the equation of AB we have $a = 2, b = 1, c = -6$ and from the coordinates of C we have $x_1 = -2, y_1 = -3$. Then the perpendicular distance between C and AB is

$$\frac{|2(-2) + 1(-3) - 6|}{\sqrt{2^2 + 1^2}} = \frac{|-4-3-6|}{\sqrt{4+1}} = \frac{|-13|}{\sqrt{5}} = \frac{13}{\sqrt{5}}$$

Example

Find the equations of the two lines that are parallel to the line 3*x* + 2*y* + 9 = 0 and $\sqrt{13}$ units from it.

Solution

Any line parallel to the line 3*x* + 2*y* + 9 = 0 will be of the form 3*x* + 2*y* + *k* = 0.

Find a point on the line 3*x* + 2*y* + 9 = 0:

Let *y* = 0

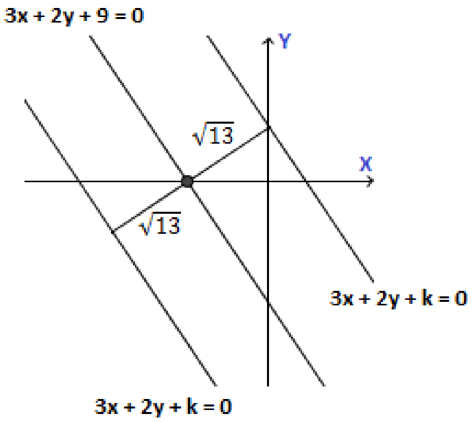
3*x* + 2(0) + 9 = 0

3*x* + 9 = 0

3*x* = - 9

x = - 3

The point (-3, 0) is on the line



The distance from the point (-3, 0) and the line

3*x* + 2*y* + *k* = 0 is $\sqrt{13}$.

$$d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

$$\sqrt{13} = \frac{|(3)(-3) + (2)(0) + k|}{\sqrt{(3)^2 + (2)^2}}$$

$$\sqrt{13} = \frac{|-9 + 0 + k|}{\sqrt{9 + 4}}$$

$$\sqrt{13} = \frac{|-9 + k|}{\sqrt{13}}$$

$$(\sqrt{13})(\sqrt{13}) = |-9 + k|$$

$$13 = |-9 + k|$$

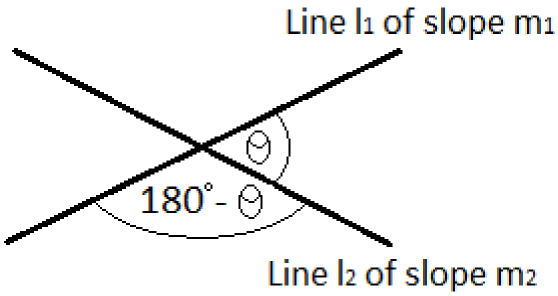
$$-13 = -9 + k \qquad 13 = -9 + k$$

$$-4 = k \qquad 22 = k$$

Therefore, the lines are:

3*x* + 2*y* – 4 = 0 3*x* + 2*y* + 22 = 0

Finding the angle between two lines:



To find the acute angle, θ, between two lines, we need the slopes, m_1 and m_2, of the two lines.

Then

$$\text{Tan}\ \theta = \pm\frac{m_1 - m_2}{1 + m_1 m_2}$$

- To find the acute angle, let

$$\text{Tan}\ \theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

The angle between perpendicular lines is 90°. Since Tan 90° is undefined this formula does not work for perpendicular lines.

- To find the obtuse angle between the lines:

  Obtuse angle = 180° - Acute angle

2013 Paper 2 Question 3

The equations of six lines are given:

Line	Equation
h	$x = 3 - y$
i	$2x - 4y = 3$
k	$y = -\frac{1}{4}(2x - 7)$
l	$4x - 2y - 5 = 0$
m	$x + \sqrt{3}y - 10 = 0$
n	$\sqrt{3}x + y - 10 = 0$

(b) Find the acute angle between the lines m and n.

Slope of m: $m_1 = -\frac{1}{\sqrt{3}}$

Slope of n: $m_2 = -\sqrt{3}$

For m_1: $y = -\frac{1}{\sqrt{3}}x + \frac{10}{\sqrt{3}}$

For m_2: $y = -\sqrt{3}x + 10$

$$\tan\theta = \pm\frac{m_1 - m_2}{1 + m_1 m_2} = \pm\frac{-\frac{1}{\sqrt{3}} + \sqrt{3}}{1 - \frac{1}{\sqrt{3}}\left(-\sqrt{3}\right)} = \pm\frac{\frac{-1+3}{\sqrt{3}}}{1+1} = \pm\frac{1}{\sqrt{3}}$$

$$\tan\theta = \frac{1}{\sqrt{3}} \quad\Rightarrow\quad \theta = 30^\circ$$

Points of Interest: By construction and geometrically

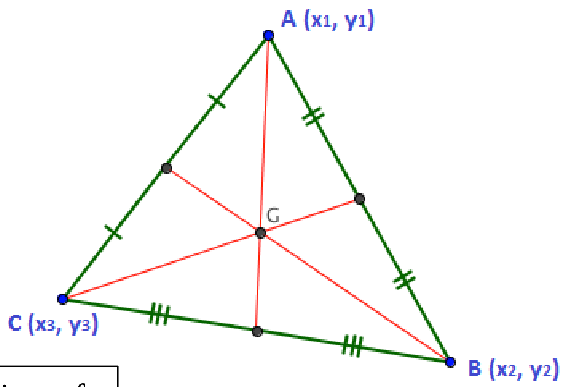
Three of the constructions from your study of Synthetic Geometry are to construct the Centroid, Circumcentre and the Orthocentre of a triangle. It is important to understand the link between these geometrical constructions and the graphing of a triangle on the coordinated plane. You should be able to use your knowledge of coordinate geometry to find the coordinates of any of these points.

In geometry, three or more lines in a plane are said to be **concurrent** if they all intersect at a single point.

The Centroid

The **median** of a triangle is the line drawn from one vertex to the midpoint of the opposite side.

The three medians of a triangle intersect at a point called the Centroid. The centroid is the centre of gravity of the triangle and it cuts the medians in the ration 2:1.
The coordinates of the Centroid are found by finding the average of the three vertices.



If $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are the vertices of a triangle, then the coordinates of the centroid, G are:

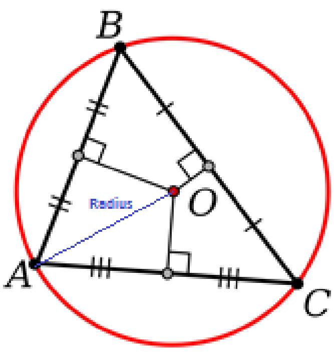
$$G = \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$$

The Circumcentre

The perpendicular bisectors of the sides of a triangle, known as the **mediators**, intersect at a point called the Circumcentre.

This point is the centre of the circumcircle, which is a circle that passes through the three vertices of the triangle.

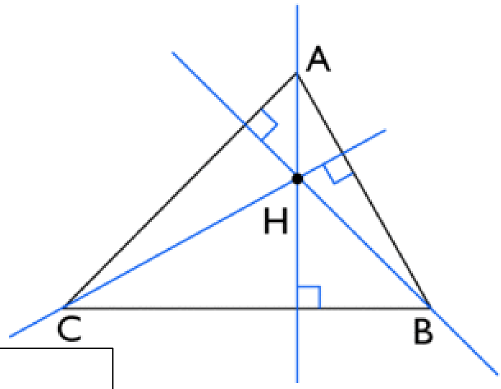
The line segment from the circumcentre to any one of the vertices of the triangle is the radius of the circumcircle.



The Orthocentre

An **altitude** is a line which passes through a vertex of a triangle and is perpendicular to the opposite side.

The point of intersection of the three altitudes of a triangle is called the Orthocentre of the triangle.



The orthocentre is:

* Inside an acute triangle
* Outside of an obtuse triangle
* At the right-angled vertex of a right-angled triangle

It's important to be aware that the circumcentre could be outside the triangle; this is perfectly normal and can be encountered at times. Students are intended to have encountered this while exploring it in class.

The following question came up in a pre-paper for the pilot schools in 2010:

Q. State the condition(s) under which the circumcentre of a triangle will lie **inside** the triangle and justify your answer.

A. The circumcentre of the circle will lie within the triangle if all the angles are acute.

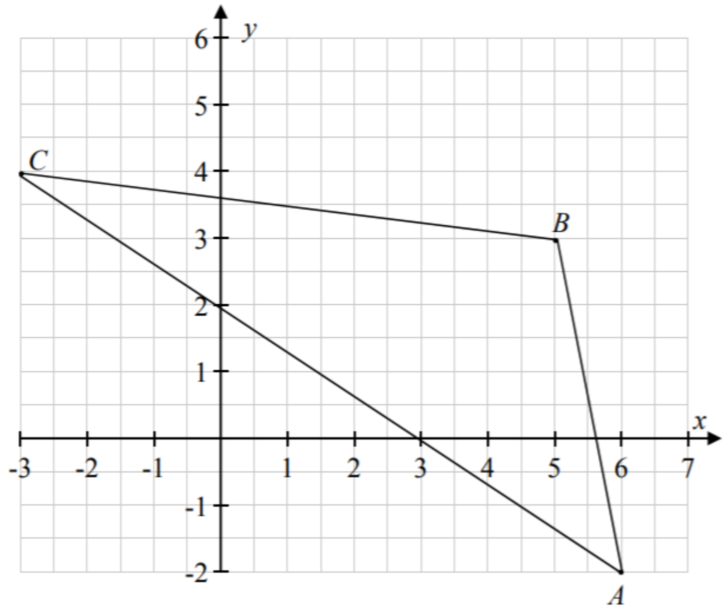
- You should be aware that if the triangle is right angled, the circumcentre is on the hypotenuse of the triangle.

- If the triangle is obtuse, the circumcentre lies outside the triangle.

2016 Paper 2 Question 1

The points $A(6, -2)$, $B(5, 3)$ and $C(-3, 4)$ are shown on the diagram.

(a) Find the equation of the line through B which is perpendicular to AC.



$$\text{Slope } AC = -\frac{2}{3}$$

$$\text{perp. slope} = \frac{3}{2}$$

$$y - 3 = \frac{3}{2}(x - 5)$$

$$3x - 2y = 9$$

(b) Use your answer to part **(a)** above to find the co-ordinates of the orthocentre of the triangle ABC.

Point of intersection of the altitudes

$$\text{Slope } AB = \frac{3+2}{5-6} = -\frac{5}{1}$$

$$\text{perp. slope} = \frac{1}{5}$$

$$y - 4 = \frac{1}{5}(x + 3)$$

$$x - 5y + 23 = 0$$

Orthocentre:

$$3x - 2y = 9 \cap x - 5y = -23$$

$$\Rightarrow y = 6 \quad x = 7$$

$$(7, 6)$$

Use simultaneous equations to find where the two lines intersect.

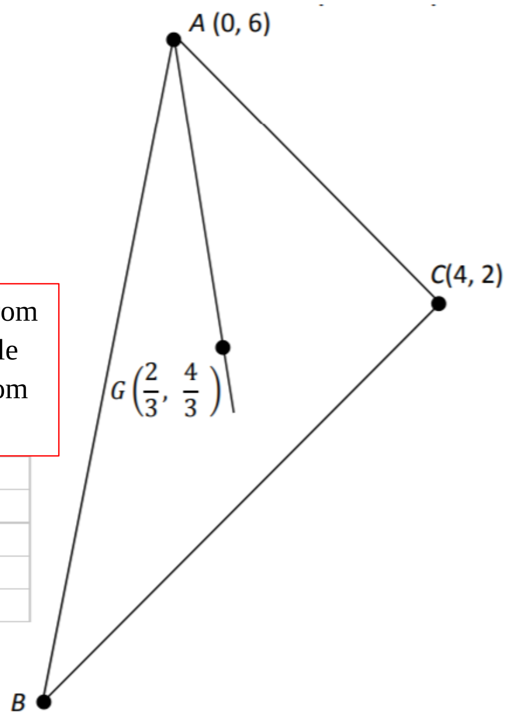
2017 Paper 2 Question 3

ABC is a triangle where the co-ordinates of *A* and *C* are (0, 6) and (4, 2) respectively.

$G\left(\frac{2}{3}, \frac{4}{3}\right)$ is the centroid of the triangle *ABC*.

AG intersects *BC* at the point *P*.

$|AG| : |GP| = 2 : 1$.



(a) Find the co-ordinates of *P*.

$$A(0,6) \to G\left(\frac{2}{3}, \frac{4}{3}\right)$$

$$\to P\left(\frac{2}{3} + \frac{1}{2}\left(\frac{2}{3}\right), \frac{4}{3} + \frac{1}{2}\left(\frac{-14}{3}\right)\right)$$

$$= \left(\frac{3}{3}, -\frac{3}{3}\right)$$

$$P = (1, -1)$$

The distance from A to G is double the distance from G to P.

(b) Find the co-ordinates of *B*.

$$C(4,2) \to P(1, \ -1) \to B(1-3, -1-3)$$

$$= (-2, \ -4)$$

$$B(x,y) \to \left(\frac{4+x}{2}, \frac{2+y}{2}\right) = (1, -1)$$

$$x = -2, \quad y = -4$$

$$B = (-2, -4)$$

(c) Prove that *C* is the orthocentre of the triangle *ABC*.

$$AC \perp BC$$

$$AC = \frac{2-6}{4-0} = -1$$

$$BC = \frac{2+4}{4+2} = 1$$

$$-1 \times 1 = -1$$

lines are perpendicular

If *C* is the orthocentre then *AC* will be perpendicular to *BC*.

2016 Paper 2 Question 1

The points $A(6, -2)$, $B(5, 3)$ and $C(-3, 4)$ are shown on the diagram.

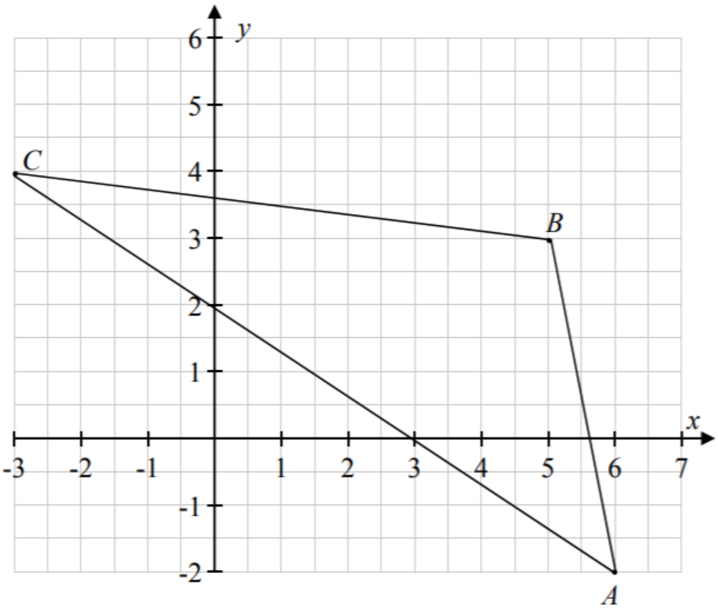
(a) Find the equation of the line through B which is perpendicular to AC.

Slope $AC = -\frac{2}{3}$

perp. slope $= \frac{3}{2}$

$y - 3 = \frac{3}{2}(x - 5)$

$3x - 2y = 9$



(b) Use your answer to part **(a)** above to find the co-ordinates of the orthocentre of the triangle ABC.

Point of intersection of the altitudes

Slope $AB = \frac{3+2}{5-6} = -\frac{5}{1}$

perp. slope $= \frac{1}{5}$

$y - 4 = \frac{1}{5}(x + 3)$

$x - 5y + 23 = 0$

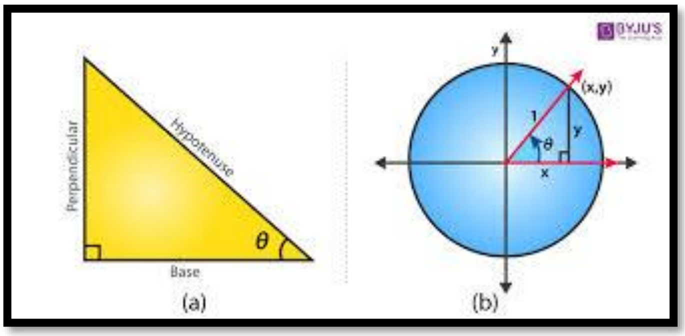
Orthocentre:

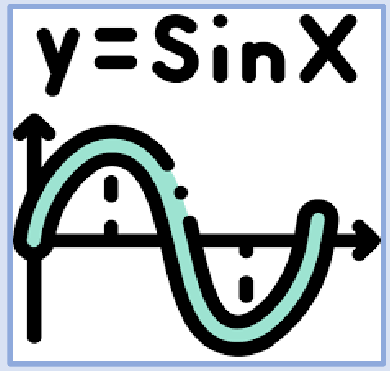
$3x - 2y = 9 \cap x - 5y = -23$

$\Rightarrow y = 6 \quad x = 7$

$(7, 6)$

Lecture Seven





Behavioral objectives

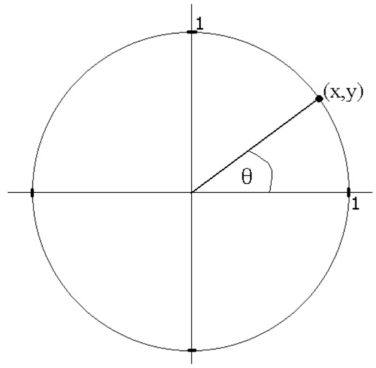
1. Define and use the six trigonometric ratios.
2. Convert between degrees and radians.
3. Apply basic trigonometric identities.
4. Use the unit circle to determine the values of trigonometric functions.

An Introduction to Trigonometry

P.Maidorn

I. Basic Concepts

The trigonometric functions are based on the unit circle, that is a circle with radius $r=1$. Since the circumference of a circle with radius r is $C=2\pi r$, the unit circle has circumference 2π.



For any point (x,y) on the unit circle, the associated angle θ can be measured in two different ways:

1. degree measure: in this case the circumference is divided into 360 equal parts, each part has measure one degree (written 1^o). A right angle, for example, is a 90^o angle. Positive angles are measured in the counter-clockwise direction.
2. radian measure: radian measure is defined as the actual length of the arc between the points (1,0) and (x,y). One entire revolution (i.e. 360^o) hence has a radian measure of 2π. A right angle (that is a quarter of one revolution) would have radian measure $\pi/2$. Note that the angle is simply denoted "$\pi/2$", not "$\pi/2$ radians".

One can easily convert between these two measures by keeping in mind that a 180^o angle (in degrees) is equivalent to a π angle (in radians). Note that angles in Calculus-related problems are usually denoted in radian measure, hence it is important to be comfortable with this measurement.

Examples:

1. A 270^o angle is 3/2 times a 180^o angle, hence in radian measure the angle would be denoted $3\pi/2$.
2. A $7\pi/5$ angle would simply have degree measure 7/5 times 180^o, i.e. 252^o.

Exercises:

Convert each angle to radians.

a) 120º b) 315º c) -420º

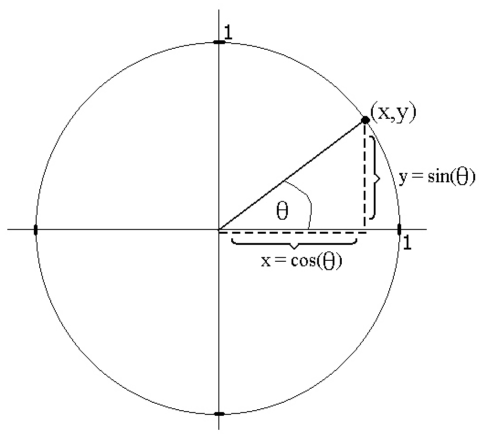
Convert each angle to degrees, to the nearest tenth of a degree.

d) $-2\pi/3$ e) 3π f) 4.52

Turning the above discussion around, each time we choose an angle θ, we find a unique point (x,y) on the unit circle. Hence both "x" and "y" can be considered functions of θ. Since these particular functions are of great importance to both pure and applied mathematics, they are given special names and symbols, and are called the trignometric functions.

Specifically:

The length "y" is called the sine of the angle θ, and is denoted by y=sin(θ).
The length "x" is called the cosine of the angle θ, and is denoted by x=cos(θ).

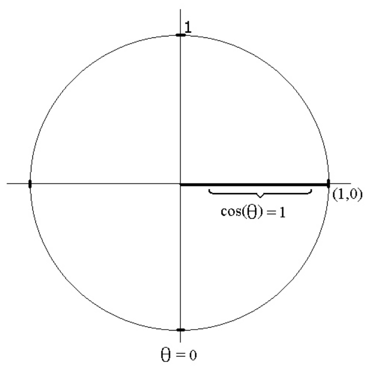


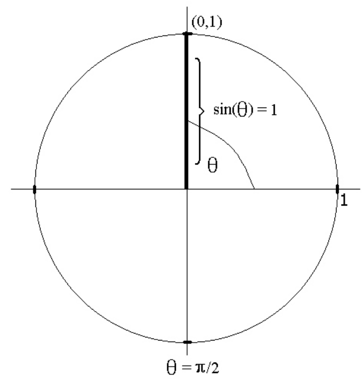
Other trigonometric functions can be calculated from the sine and cosine functions: the tangent of θ is defined as tan(θ)=sin(θ)/cos(θ) (or y/x), the secant of θ is defined as sec(θ)=1/cos(θ), the cosecant of θ is defined as csc(θ)=1/sin(θ), and the cotangent of θ is defined as cot(θ)=cos(θ)/sin(θ).

II. Calculating Trigonometric Functions of Special Angles

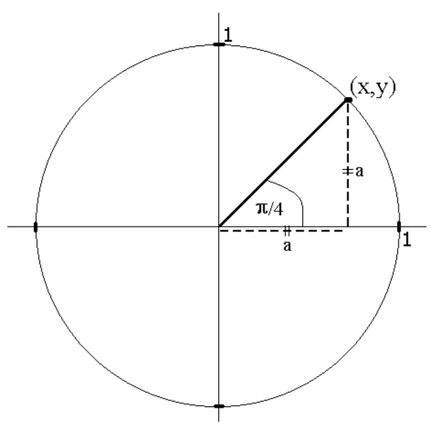
The first question that arises is how to calculate the sine or cosine of a given angle, that is how to find the lengths "x" (the cosine) and "y" (the sine) on the unit circle associated with a given angle.

We will begin to answer this question by looking at the angles 0º and 90º (or π/2). First, draw the unit circle, and on it indicate the angle θ=0 as well as the point (x,y) that is associated with that angle. If the angle is θ=0, the point (x,y) lies on the x-axis, that is x=1, and y=0 (remember that the radius of the circle is r=1). Hence cos(0)=1 and sin(0)=0. Similarly, the angle θ=π/2 is associated with the point (x,y)=(0,1). Therefore cos(π/2)=0 and sin(π/2)=1 (see diagrams).





Let's examine the angle θ=π/4 (or 45º) next.



Note that a right-angle triangle is formed, with a hypotenuse of length 1, and two adjacent sides of equal length, that is x=y. Let's denote that length "a". By the Pythagorean theorem, we have

$$a^2+a^2=1^2,$$

which we can solve for $a=\sqrt{½}$ or equivalently $a=\sqrt{2}/2$. Hence both "x" and "y" are equal to $\sqrt{2}/2$, and we have found that both $\sin(\pi/4)=\sqrt{2}/2$ and $\cos(\pi/4)=\sqrt{2}/2$.

One can also find trigonometric values for the angles $\theta=\pi/6$ (or 30º) and $\theta=\pi/3$ (or 60º). This set of angles is sometimes called the "special" angles, and their associated sine and cosine values are listed in the table below:

θ	**sin(θ)**	**cos(θ)**
0	0	1
$\pi/6$	½	$\sqrt{3}/2$
$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$
$\pi/3$	$\sqrt{3}/2$	½
$\pi/2$	1	0

We can immediately use these values to calculate other trigonometric functions of these special angles.

Examples:

1. Since $\tan(\theta)=y/x$, i.e. $\tan(\theta)=\sin(\theta)/\cos(\theta)$, we simply divide $\sin(\pi/3)$ by $\cos(\pi/3)$ to find that $\tan(\pi/3) = \sqrt{3}$.
2. Similarly $\csc(\pi/6) = 1 / \sin(\pi/6)$, that is $\csc(\pi/6) = 2$.

Note that not all trigonometric functions are defined for all angles. For example, the tangent of $\theta=\pi/2$ does not exist, since here the denominator is equal to zero.

Exercises:

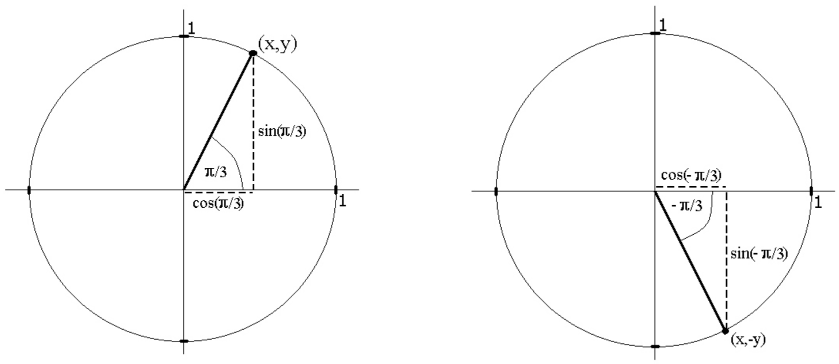
Calculate:

a) $\sec(\pi/3)$ b) csc(45º) c) cot(0)

III. Other Angles

Once you know how to find the trigonometric functions for the above special angles, it is important to learn how to extend your knowledge to any angle that is based on one of 0, $\pi/6$, $\pi/4$, $\pi/3$ or $\pi/2$, such as for example $2\pi/3$, $-\pi/6$, $7\pi/4$, $-5\pi/2$, and others.

Let's examine the angle $\theta=-\pi/3$. Clearly, it is somehow related to the angle $\pi/3$. Draw a unit circle for both angles side-by-side, and indicate the sine and cosine on them:

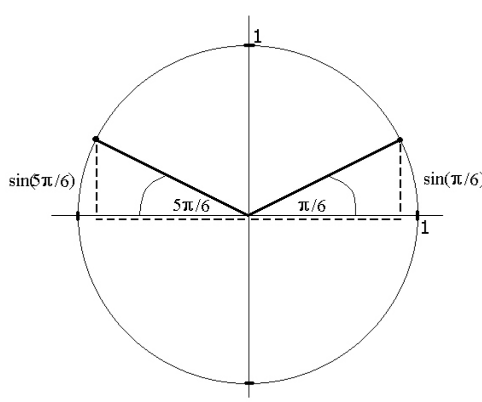


Clearly, the right-angle triangles that are formed are identical, except that they are mirror images of each other. The cosine ("x") in both cases is the same, hence we know that $\cos(-\pi/3)$ is identical to $\cos(\pi/3)$, that is $\cos(-\pi/3)=½$. The sine ("y") is the same length, but has opposite sign (it is negative). Since $\sin(\pi/3)=\sqrt{3}/2$, then $\sin(-\pi/3)=-\sqrt{3}/2$.

In each of the examples below, proceed with the same method. It is imperative that you draw the unit circle each time until you become comfortable with these types of questions.

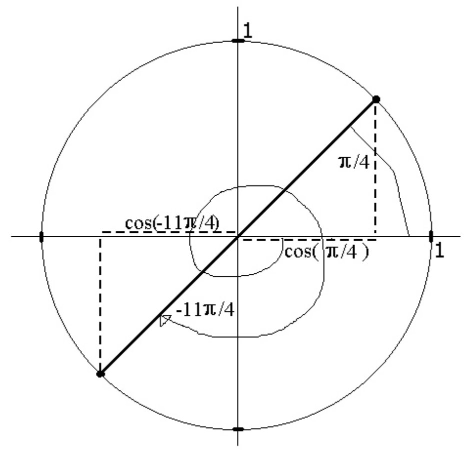
Examples:

1. Find $\sin(5\pi/6)$. The triangle formed by $\theta=5\pi/6$ is identical to that formed by $\theta=\pi/6$, except that it is reflected across the y-axis. The sine of $5\pi/6$ is hence the same length, and has the same sign, as the sine of $\pi/6$. Therefore $\sin(5\pi/6)=½$.



2. Find sec($-11\pi/4$). To find the secant of an angle, remember to find the cosine first and take its reciprocal. The cosine of $-11\pi/4$ is related to the cosine of $\pi/4$. The angle $-11\pi/4$ is reached by completing one entire clockwise revolution (which equals -2π, or $-8\pi/4$) and then adding another $-3\pi/4$. Compare the angle $-3\pi/4$ to $\pi/4$. The cosine is the same length in each case, but has opposite sign. Since $\cos(\pi/4) = \sqrt{2}/2$, $\cos(-11\pi/4) = -\sqrt{2}/2$, and hence

$\sec(-11\pi/4) = -2/\sqrt{2} = -\sqrt{2}$.



Exercises:

Calculate

a) $\sin(-\pi/4)$	b) $\sin(5\pi/2)$	c) $\cos(11\pi/6)$
d) $\cos(13\pi/4)$	e) $\cos(19\pi/6)$	f) $\sin(-510^\circ)$

IV. Using the Calculator

If you need to calculate the sine, cosine, or tangent of an angle other than the ones discussed above, you may need to use your calculator. Consult your calculator manual, if necessary, on how to use the trigonometric functions.

You should be aware of two facts:

1. In most cases, your calculator will not give you exact answers, but rather decimal approximations. For example, your calculator will tell you that the sine of a 45° angle is approximately .70710678, rather than giving you the exact answer $\sqrt{2}/2$.

2. You need to set your calculator to the appropriate angle measure, degrees or radians. Otherwise, your calculator might return the sine of the angle $\theta \approx 3.1415...$ degrees, when you enter "sin(π)" looking for the sine of $\theta=180^\circ$.

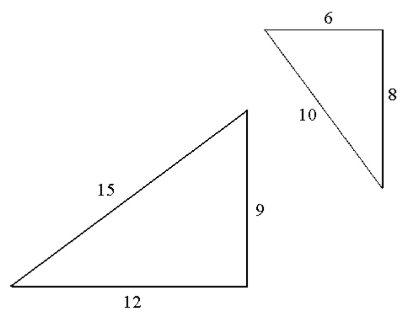
Exercises:

Use your calculator to compute, rounding to four decimal places:

a) $\sin(53.7^o)$ b) $\cos(11\pi/7)$ c) $\csc(-32^o)$

V. Right-Triangle Applications

In order to apply the trigonometric functions based on the unit circle to right triangles of any size, it is important to understand the concept of similar triangles. Two triangles are said to be similar if the ratio of any two sides of one triangle is the same as the ratio of the equivalent two sides in the other triangle. As a result, similar triangles have the same "shape", but might differ in size. For example, the sides in the triangles below have the same ratios to each other.



Similar Triangles

Consider the right triangle inscribed in the unit circle associated with an angle θ. We can calculate the length of the side adjacent to the angle θ (i.e. cos(θ)) and the length of the side opposite the angle θ (i.e. sin(θ)). Since the unit circle has radius one, the hypotenuse of these triangles is always equal to one.

If we were given a triangle with identical angle θ but with a hypotenuse twice the length, each of the other sides would be twice the length as well, as the triangles are similar. We can use this fact to now compute side-lengths of any right triangle, if the angle θ and one side-length are known. In general, we have

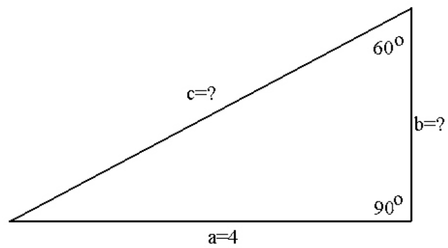
$\sin(\theta)$ = opposite / hypotenuse and $\cos(\theta)$ = adjacent / hypotenuse

and since $\tan(\theta)=\sin(\theta)/\cos(\theta)$, we have

$$\tan(\theta) = \text{opposite / adjacent.}$$

Examples:

1. Find all missing sides and angles in the given triangle.



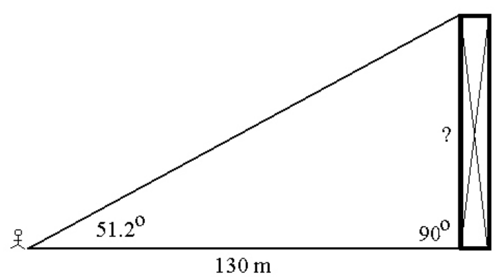
First, given that the sum of all angles in any triangle must equal 180º (or π), the missing angle measures 30º. We know the length of the opposite side of angle θ, hence we can use the sine to find the length of the hypotenuse. Since

$\sin(60^\circ) = \sqrt{3}/2$ and $\sin(\theta) = \text{opposite} / \text{hypotenuse}$

we have $\sqrt{3}/2 = 4 / c$, which we can solve for $c = 8/\sqrt{3}$ or $c = 8\sqrt{3} / 3$.

Finally use, for example, the Pythagorean theorem to find that $b = 4\sqrt{3} / 3$.

2. To find the height of a building, a person walks to a spot 130m away from the base of the building and measures the angle between the base and the top of the building. The angle is found to be 51.2º. How tall is the building?



We know the length of the adjacent side of the right triangle that is formed, and wish to find the length of the opposite side. The quickest method to do so is to use the tangent, since $\tan\theta = \text{opposite} / \text{adjacent}$. Using the calculator, $\tan(51.2^\circ) \approx 1.2437$. Hence the height of the building is

$$h = 130 \tan(51.2^\circ) \approx 161.7 \text{ metres.}$$

Exercises:

a) A ladder is leaning against the side of the building, forming an angle of 60º with the ground. If the foot of the ladder is 10m from the base of the building, how far up does the ladder reach, and how long is the ladder?

b) A person stationed on a 40m tall observation tower spots a bear in the distance. If the angle of depression (that is, the angle between the horizontal and the line of sight) is 30º, how far away is the bear, assuming that the land surrounding the tower is flat?

c) A 12m tall antenna sits on top of a building. A person is standing some distance away from the building. If the angle of elevation between the person and the top of the antenna is 60º, and the angle of elevation between the person and the top of the building is 45º, how tall is the building and how far away is the person standing?

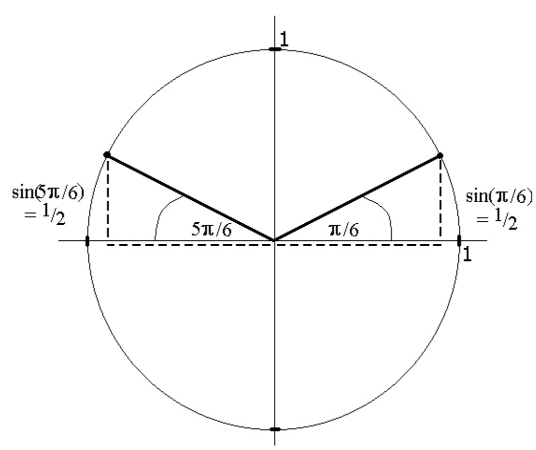
VI. Simple Trigonometric Equations

Given a value for the angle "x", we now know how to calculate sin(x), cos(x), etc. We need to proceed more carefully if however we wish to solve for the angle "x" given the value of sin(x) or cos(x).

Example: Find all values for "x" (in radian measure) such that sin(x)=½.

One such angle "x" can be immediately found from our table of special angles. Since sin(π/6)=½ , we know that one possible x-value is x=½. However, this is not the only possibility.

Consider the unit circle for $\theta_1=\pi/6$. Is it possible to find another angle such that the sine value (i.e. the height "y") is identical to that of $\theta_1=\pi/6$? The answer is 'yes'. Another such angle is on the other side of the y-axis. Since this angle is π/6 away from the angle π (180º), it measures $\theta_2 = \pi-\pi/6 = 5\pi/6$. Hence sin(5π/6) is also equal to ½ (check this on your calculator).



Are there any other such angles? By remembering that we can always add or subtract one complete revolution (2π) from an angle to end up in the same position, we can in fact generate infinitely many such angles. For example, $\theta_3 = \pi/6 + 2\pi = 13\pi/6$ is another such angle, as is $\theta_4 = 5\pi/6 - 4\pi = -19\pi/6$.

Hence all angles $\theta = \pi/6 + 2k\pi$ and $\theta = 5\pi/6 + 2k\pi$ for any $k \in I$ (that is k can be any integer ...-3,-2,-1,0,1,2,3,...) are solutions to the equation $\sin(x)=½$.

Exercises:

Find all "x" such that

a) $\cos(x) = \sqrt{3}/2$ b) $\cos(x) = -\sqrt{3}/2$ c) $\sin(x) = -1$

d) $\sin(x) = -½$ e) $\cos(x) = -½$ f) $\tan(x) = 1$

VII. Trigonometric Identities

There are several trigonometric identities, that is equations which are valid for any angle θ, which are used in the study of trigonometry.

From the unit circle, we have already seen that the cosine of an angle is identical to the cosine of the associated negative angle, that is

$$\cos(-\theta)=\cos(\theta) \qquad \text{for any angle } \theta. \tag{1}$$

Similarly,

$$\sin(-\theta)=-\sin(\theta) \qquad \text{for any angle } \theta. \tag{2}$$

For example, $\sin(\pi/6) = ½$ and $\sin(-\pi/6) = -½$.

Also from the unit circle, which has equation $x^2+y^2=1$, we can substitute $x=\cos(\theta)$ and $y=\sin(\theta)$ to obtain the identity

$$\sin^2(\theta)+\cos^2(\theta)=1, \qquad \text{for any angle } \theta. \tag{3}$$

Another useful trigonometric identity concerns the sum of two angles A and B. We have:

$$\sin(A+B) = \sin(A)\cos(B) + \sin(B)\cos(A) \tag{4}$$

and

$$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B) \tag{5}$$

for any angles A and B.

Note that you cannot simply "distribute" the sine through a sum. It is false to state that, for example, $\sin(A+B)=\sin(A) + \sin(B)$.

The above five identities can be used to derive many other useful identities, which then no longer need to be memorized.

Examples:

1. Dividing equation (3) by $\cos^2(\theta)$, we obtain the identity

$$1 + \tan^2(\theta) = \sec^2(\theta).$$

2. Identities involving the difference of two angles A-B can be obtained by combining equation (4) or (5) with equations (1) and (2). As an example,

$\sin(A-B) = \sin(A+(-B))$

$= \sin(A)\cos(-B) + \sin(-B)\cos(A)$ — using (4)

$= \sin(A)\cos(B) - \sin(B)\cos(A)$ — using (1) and (2) on the left and right term respectively.

3. The double-angle formula $\cos(2\theta) = 2\cos^2(\theta) - 1$ can be derived using equation (5) and setting both $A=\theta$ and $B=\theta$:

$\cos(2\theta) = \cos(\theta + \theta)$

$= \cos(\theta)\cos(\theta) - \sin(\theta)\sin(\theta)$ — using (5)

$= \cos^2(\theta) - \sin^2(\theta)$

Now, re-arrange (3) to obtain $\sin^2(\theta)=1-\cos^2(\theta)$ and substitute:

$\cos(2\theta) = \cos^2(\theta) - (1-\cos^2(\theta))$

$= 2\cos^2(\theta) - 1.$

4. Find all "x" such that $2\sin^2(x) + \cos(x) = 1$.

To solve this trigonometric equation, we will first need to simplify it using the trigonometric identities. Using (3) we can substitute $\sin^2(x)=1-\cos^2(x)$ and obtain an equation that only involves cosines:

$$2 - 2\cos^2(x) + \cos(x) = 1$$

or

$$2\cos^2(x) - \cos(x) - 1 = 0.$$

The above is a quadratic equation in cos(x), and can be factored

$$(2\cos(x)+1)(\cos(x)- 1) = 0.$$

As a result, we now need to find all "x" such that either

$2\cos(x) + 1 = 0$, i.e. $\cos(x)=-\frac{1}{2}$,

or $\cos(x)-1 =0$, i.e. $\cos(x)=1$.

From the unit circle, $\cos(x)=1$ when $x= 0 +2k\pi,\ k\in I$, i.e. $x=2k\pi,\ k\in I$

Further, $\cos(x)=-\frac{1}{2}$ when $x=2\pi/3+2k\pi$ or $x=4\pi/3+2k\pi,\ k\in I$.

The solution to the given equation is hence the set of all x-values listed above.

Exercises:

Derive the following equations using equations (1) through (5) only:

a) $\cos(A-B) = \cos(A)\cos(B) + \sin(A)\sin(B)$

b) $\sin(2A) = 2\sin(A)\sin(B)$

c) $\sin(A)\sin(B) = -\frac{1}{2} [\cos(A+B) - \cos(A-B)]$

Solve the following equations for "x":

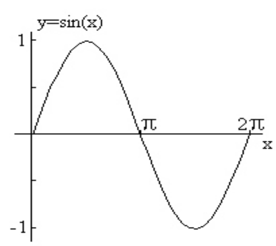
d) $1 - \cos(2x) = 2\sin(x)$

e) $\sin^2(5x^3-2x^2+1) = 1 - \cos^2(5x^3-2x^2+1)$

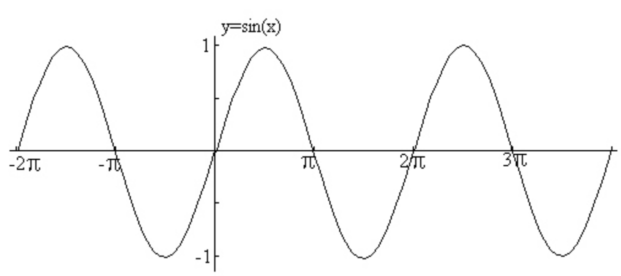
VIII. Graphing Trigonometric Functions

Recall that a function y=f(x) is a rule of correspondence between the independent variable ("x") and the dependent variable ("y"), such that each x-value is associated with one and only one y-value. Since each angle "x" produces only one value for sin(x), cos(x), etc., the relationships y = sin(x), y = cos(x), etc. are functions of "x".

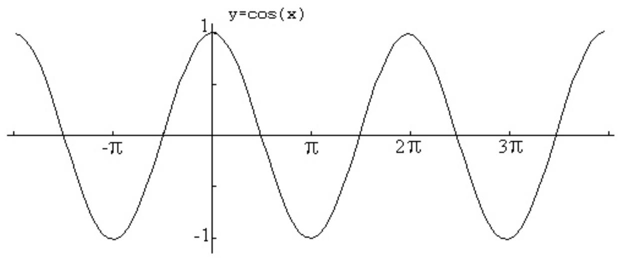
To obtain the graph of the function y = sin(x), trace how the height of the triangle inscribed in the unit circle changes as the angle "x" gradually moves from x=0 to x=2π. For example, the sine graph will start at (0,0) since the sine of zero is zero, obtain a maximum value at (π/2, 1), since the sine of π/2 equals one, then decrease towards (π,0).



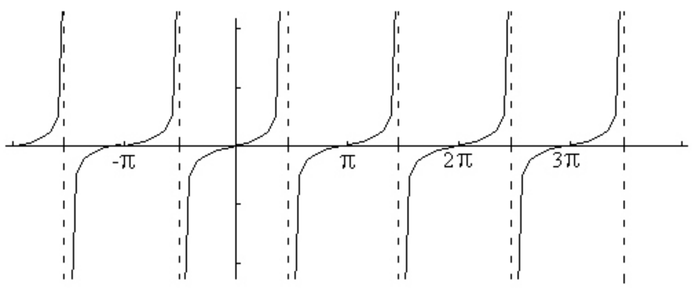
Since the graph of y = sin(x) will repeat as we complete more than one revolution, we can now obtain the complete graph of the real valued function y = sin(x).



The graph of y = cos(x) is obtained in a similar fashion.



To obtain the graph of y = tan(x), divide sin(x)/cos(x) as before. Note again that tan(x) does not exist for values of $x=\pi/2 + k\pi$, $k \in I$.

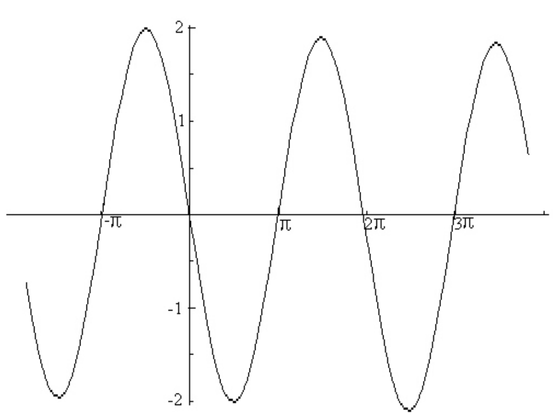


You can now use the techniques for shifting and scaling function graphs to obtain the graphs for any trigonometric function.

Examples:

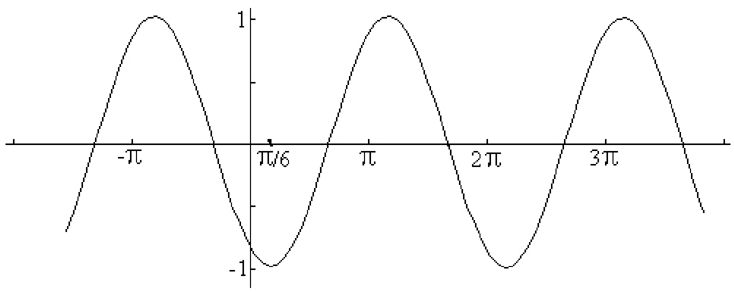
1. Graph $y = 2\sin(x+\pi)$.

This graph is identical to that of y=sin(x), except it is shifted to the left by π units, and scaled vertically by a factor of 2.



2. Graph $y = -\cos(x-\pi/6)$.

This graph is obtained by shifting the graph of $y=\cos(x)$ to the right by $\pi/6$ units, then flipping it across the x-axis.



<u>Exercises:</u>

Sketch graphs for the given functions:

a) $y = \cos(x+\pi/6) - ½$

b) $y = ½ \sin(x-\pi/2) + 1$

Answers to Exercises

I. a) $2\pi/3$ b) $7\pi/4$ c) $-7\pi/3$ d) -120^{o} e) 540^{o} f) 259.0^{o}

II. a) 2 b) $\sqrt{2}$ c) Does not exist

III. a) $-\sqrt{2}/2$ b) 1 c) $\sqrt{3}/2$ d) $-\sqrt{2}/2$ e) $-\sqrt{3}/2$ f) $-½$

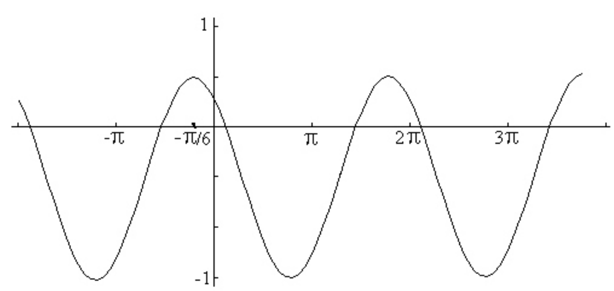
IV. a) .8059 b) .2225 c) -1.8871

V. a) The ladder is 20m long and reaches up approximately 17.32m.
b) The bear is approximately 69.28m away.
c) The height of the building and the distance of the observer are both about 20.49m.

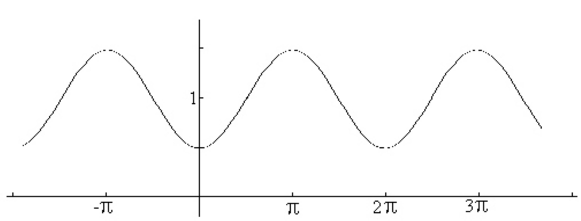
VI. a) $\pi/6 + 2k\pi$ and $11\pi/6 + 2k\pi$
b) $5\pi/6 + 2k\pi$ and $7\pi/6 + 2k\pi$
c) $3\pi/2 + 2k\pi$
d) $7\pi/6 + 2k\pi$ and $11\pi/6 + 2k\pi$
e) $2\pi/3 + 2k\pi$ and $4\pi/3 + 2k\pi$
f) $\pi/4 + k\pi$

VII. d) $k\pi$ and $\pi/2 + 2k\pi$
e) all x-values

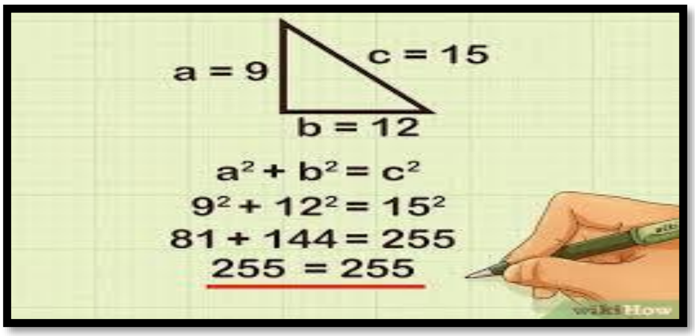
VIII. a)



b)

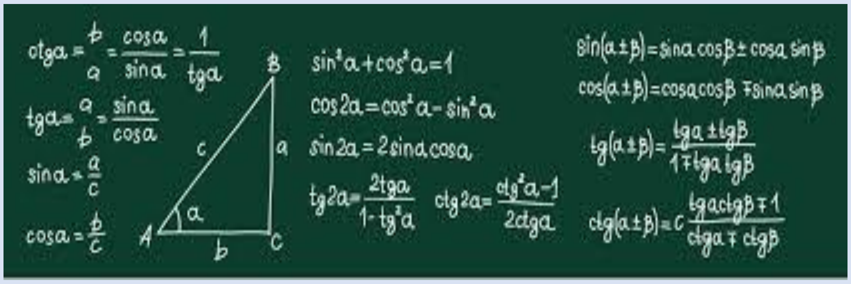


Lecture Eight



Behavioral objectives

1. Identify sides and angles of a triangle using standard notation.
2. Apply the Law of Sines and Law of Cosines to solve triangles in various configurations.
3. Determine unknown sides and angles when provided with three known values.
4. Solve real-world problems involving triangle measurements



Solving Triangles

To solve a given triangle means to determine all its unknown angles and sides. There is a standard way of labelling triangles which makes it relatively easy to describe what has to be calculated. The vertices are labelled A, B and C as shown in figure (i).

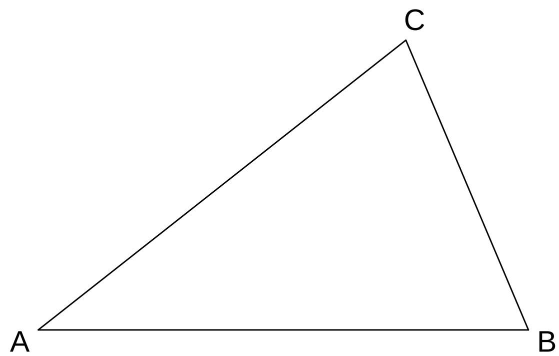


Figure (i)

The order is not important.

The angles at the vertices A, B and C should be denoted by $\angle A$, $\angle B$ and $\angle C$ respectively. Thus in figure (ii) below, we should write $\angle A = 41°$, $\angle B = 78°$ and $\angle C = 61°$.

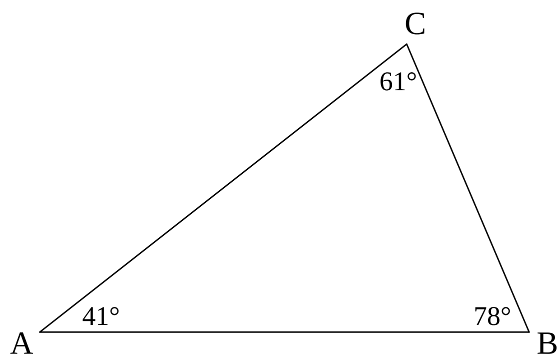


Figure (ii)

However, the symbol $\angle$ is so cumbersome, we are forced to drop it. Thus we simply write $A = 41°$, $B = 78°$ and $C = 61°$. It is understood that the letters refer to the sizes of the angles, NOT the vertices.

We use lower case letters to denote the lengths of the sides. It is agreed to denote the length of the side opposite angle A by a, the length of the side opposite angle B by b and that of the side opposite angle C by c. This is shown on figure (iii) below

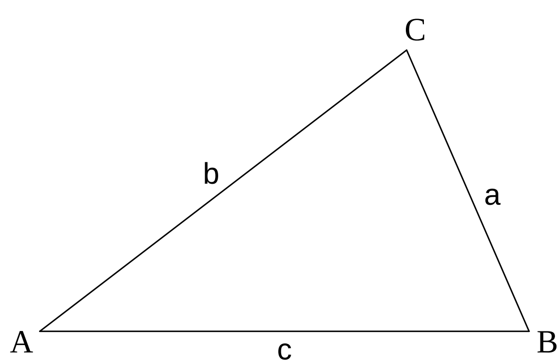
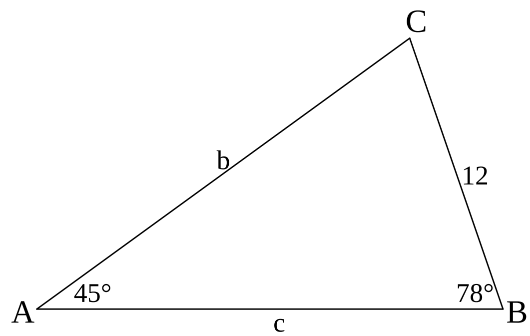


Figure (iii)

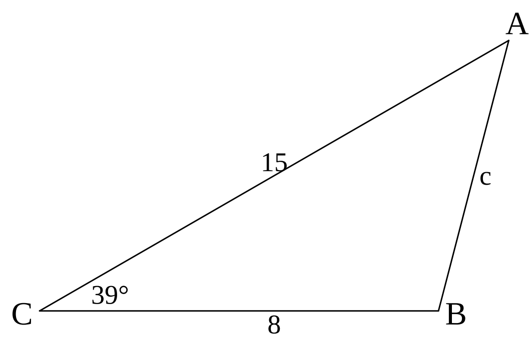
Before proceeding to solve a given triangle ABC, one must be furnished with at least three of the six variables $\angle A$, $\angle B$, $\angle C$, a, b, and c. There are a total of four different possibilities, (we are dropping the symbol $\angle$), and they are:

1. Two angles and one of the sides are given, (any two angles and any one side). Example: $A = 45°$, $B = 78°$ and $a = 12$ cm.



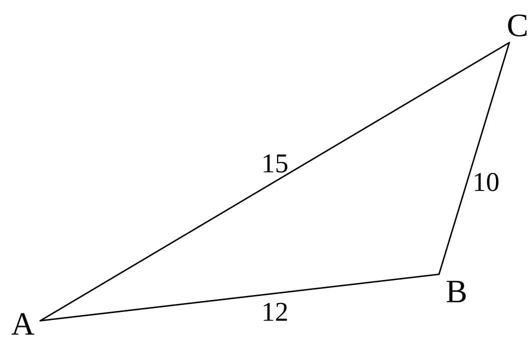
   We have to determine C, b and c.

2. Two sides and the angle between them are given. Example: $C = 39°$, $a = 8$ inches and $b = 15$ inches.



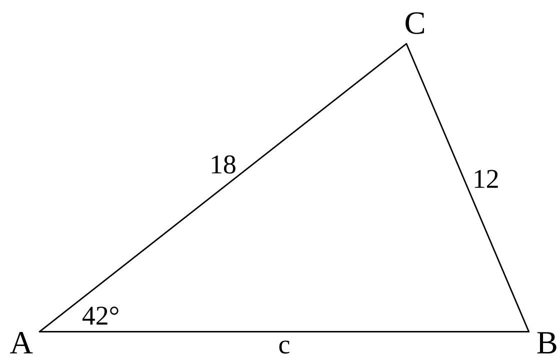
   We have to determine c, A and B.

3. Three sides are given. Example $a = 10$ cm, $b = 15$cm and $c = 12$ cm.



   We have to find the three angles A, B and C.

4. Two sides are given and an angle opposite one of the given sides is also given. Example: $A = 42°$, $b = 18$ cm and $a = 12$ cm.



We have to find B, C and c.

We cannot solve a triangle given only its three angles because there are many triangles, with different sizes, that have the same three angles.

The Law of Sines

The law of sines states that if a triangle is labelled in the standard way as shown in figure (iii) above then

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

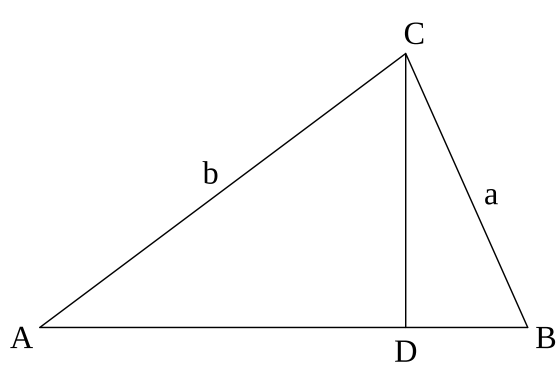
These three equations are usually combined into one expression as

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Actually we should write

$$\frac{a}{\sin \angle A} = \frac{b}{\sin \angle B} = \frac{c}{\sin \angle C}$$

but, as we have pointed out above, the cumbersome symbol $\angle$ is ignored for convenience. To derive the law, drop a perpendicular CD from C to AB as shown in figure (iii) below.



(iv)

Since angle ADC is a right angle, $\sin A = \dfrac{CD}{b}$ and $\sin B = \dfrac{CD}{a}$. It follows that $b \sin A = a \sin B$ because they are each equal to CD. This implies that

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

If you drop a perpendicular from B to AC then you obtain

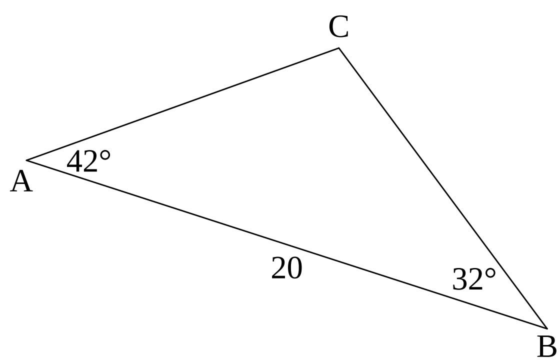
$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

Since $\dfrac{b}{\sin B}$ and $\dfrac{c}{\sin C}$ are equal to the same quantity $\dfrac{a}{\sin A}$, the third equation $\dfrac{b}{\sin B} = \dfrac{c}{\sin C}$ follows.

The Law of Sines may be used to solve a triangle in which two angles and one side are given. We call this solving a **Side, Angle, Angle** problem, abbreviated to **SAA** problem.

Example 1 *To solve the triangle ABC with $A = 42°$, $B = 32°$ and $c = 20$ cm.*

It is a good idea to draw a realistic, big diagram.



We have to find C, a and b. Since the angles of a triangle add up to 180°, C is easy to find. It is

$$180° - 42° - 32° \text{ degrees.}$$

Thus $C = 106°$.

It remains to determine a and b. Since we know c and C, we can obtain a from

$$\frac{a}{\sin 42°} = \frac{c}{\sin C} = \frac{20}{\sin 106°}$$

The result is $a = \dfrac{20 \sin 42°}{\sin 106°} = 13.9$ to 1 dec. pl.

We get b in a similar way:

$$\frac{b}{\sin 32°} = \frac{20}{\sin 106°}$$

Therefore $b = \dfrac{20 \sin 32°}{\sin 106°} = 11.2$ to 1 dec. pl. Now the triangle is solved since we know all its angles and sides.

Exercise 2 *Solve the following triangles ABC:*

1. $B = 44°$, $C = 65°$ *and* $c = 28$ *cm.*
2. $C = 35°$, $A = 27°$ *and* $b = 120$ *m.*
3. $A = 90°$, $b = 7$ *m and* $c = 24$ *m. (Hint: This is a triangle with aright angle. Use the Pythagorean theorem to get a then use the law of sines to determine B or C.)*
4. $B = 90°$, $b = 20$ *cm and* $a = 12$ *cm.*
5. $B = 20°$, $C = 50°$ *and* $a = 45$ *cm.*
6. $C = 60°$, $A = 44°$ *and* $b = 450$ *m.*

The Law of Cosines

Label a triangle in the standard way. Then the law of cosines simply states that

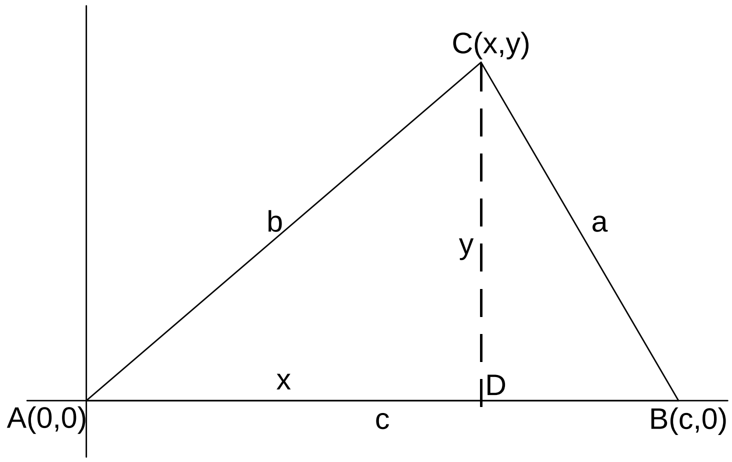
$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$b^2 = c^2 + a^2 - 2ac\cos B$$

$$c^2 = a^2 + b^2 - 2ab\cos C$$

The choice of which one of the three to use is dictated by what is given.

For a proof, consider the triangle ABC shown below. Let C have coordinates (x, y).



We note that:

1. AB has length c, therefore DB has length $c - x$
2. $x = b\cos A$ and $y = b\sin A$. It follows that DB has length $(c - b\cos A)$

Now we apply the Pythagorean theorem to triangle BCD to get

$$a^2 = (b\sin A)^2 + (c - b\cos A)^2$$

Expand and simplify. You should end up with

$$a^2 = b^2 + c^2 - 2bc\cos A$$

The Law of cosines can be used to solve a triangle in which the lengths of all the three sides are given. We call this solving a **Side, Side, Side** problem, abbreviated to **SSS**. It can also be used to solve a triangle in which two sides and the angle between them are given. We call this a **Side, Angle, Side** problem, abbreviated to **SAS**.

Example 3 *To solve the triangle ABC with $a = 7$ cm, $b = 11$ cm and $c = 6$ cm.*

This is a SSS problem therefore the unknowns are the three angles. It is best to start by calculating the largest angle, which is B (the angle facing the longest side). Therefore we start from the equation

$$b^2 = a^2 + c^2 - 2ac\cos B$$

Substituting the given lengths into the equation gives

$$121 = 49 + 36 - 2\,(7)\,(6)\cos B \qquad OR \qquad 121 = 85 - 84\cos B$$

The only unknown is $\cos B$ and when we solve for $\cos B$ we get

$$\cos B = -\frac{36}{84}$$

Therefore $B = \cos^{-1}\left(-\frac{36}{84}\right)$. A calculator gives $B = 115.5°$ to one decimal place.

The same procedure may be used to calculate A. This time we start from

$$a^2 = b^2 + c^2 - 2bc\cos A$$

Substituting the given lengths gives

$$49 = 121 + 36 - 2\,(11)\,(6)\cos B \qquad OR \qquad 49 = 157 - 132\cos A$$

Therefore

$$\cos A = \frac{108}{132}$$

hence $A = \cos^{-1}\left(\frac{108}{132}\right) = 35.1°$.

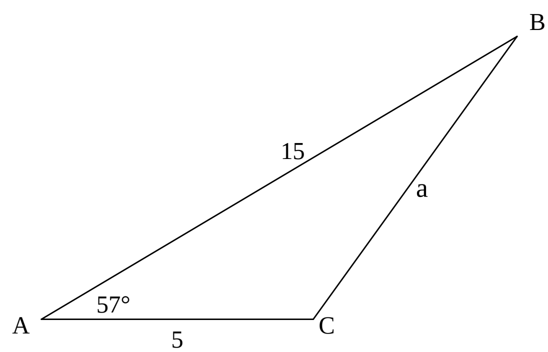
Since the angles of a triangle add up to 180°, we may get the third angle by subtraction, (it is simpler than resorting to $c^2 = a^2 + b^2 - 2ab\cos C$*):*

$$C = 180 - 115.5° - 35.1° = 29.4°$$

The triangle is solved.

Example 4 *To solve the triangle ABC with* $A = 57°$, $b = 5$ *inches and* $c = 15$ *inches and round off the resulting figures to 1 decimal place.*

A figure showing the given information is given below.



This is a SAS problem and we have to determine a, angle B and angle C. We start by determining a. This is obtained from

$$a^2 = b^2 + c^2 - 2bc\cos A$$

Substituting the given sides and angle gives

$$a^2 = 25 + 225 - 2\,(5)\,(15)\cos 57°$$

Therefore $a = \sqrt{250 - 150\cos 57°} = 12.973$ *inches which we may round off to 13.0 cm 1 decimal places.*

It remains to determine one of the angles then use a subtraction to get the remaining one. Another use of the Law of Cosines to determine C we would start from

$$c^2 = a^2 + b^2 - 2ab\cos C$$

Substituting the given lengths gives

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{13^2 + 5^2 - 15^2}{2\,(13)\,(5)}$$

Solve to get $C = 103.8°$ *to 1 decimal place. The third angle is* $B = (180 - 103.8 - 57) = 19.2$ *degrees.*

One may also use the Law of Sines, after getting a but one has to be cautious because given a positive real number y, there are two angles x between 0° and 180° such that $\sin x = y$. *With this in mind, if we choose to use the Law of Sines to find C in this problem, we would start from*

$$\frac{\sin C}{15} = \frac{\sin A}{13} \quad OR \quad \sin C = \frac{15 \sin 57}{13}$$

Therefore $\sin C = 0.9677$ *and a calculator gives* $C = 75.4°$, *(rounded off to 1 decimal place). Actually* $C = 75.4°$ *or* $C = 180° - 75.4° = 104.6°$ *because there are two angles between between 0° and 180° whose sine is* 0.9677. *Clearly, C cannot be* $75.4°$ *because it is the largest angle of the triangle, (since it faces the longest side of the triangle). Therefore we must take* $C = 104.6°$. *Then* $B = 180° - 104.6° - 57° = 18.4°$. *(The difference in the two sets of answers is a result of rounding off errors. We may reduce it by keeping more decimal places in our calculations.)*

Exercise 5

1. *Use the law of cosines to solve the triangle ABC with* $a = 10$ *cm,* $b = 12$ *cm, and* $c = 15$ *cm.*
2. *Use the law of cosines to solve the triangle ABC with* $a = 8$ *cm,* $b = 12$ *cm, and* $C = 48°$.

The following figures are referred to in Questions (3) and (4)

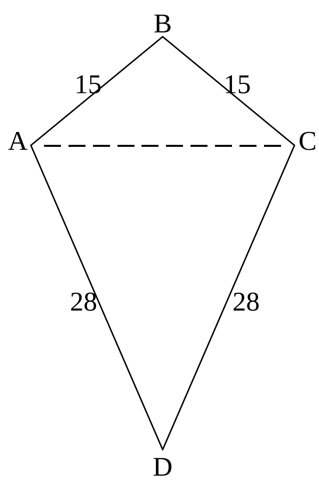


Figure 1

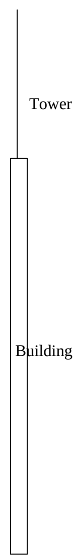


Figure 2

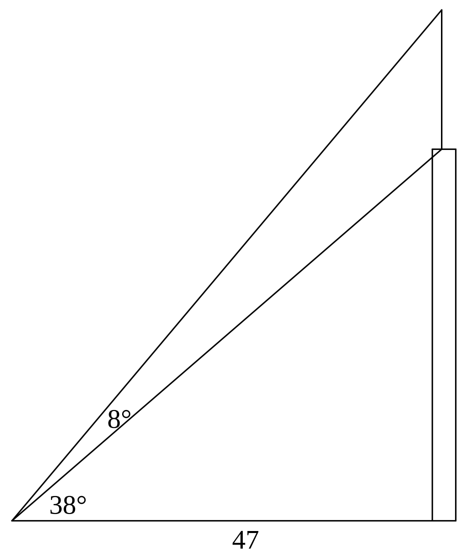


Figure 3

3. *Figure 1 above shows a kite ABCD. The lengths are all in inches. If AC has length 20 inches, calculate the angles ABC, ADC and BCD.*
4. *A transmission tower is on top of a vertical building as shown in Figure 2 above. The angles of elevation of the bottom and top of the tower were measured from a point 47 meters from the bottom of the building and the results are as shown in Figure 3. Calculate the height of the transmission tower.*

The Side, Side, Angle problem

In the **Side, Side, Angle** problem, you are given two sides of a triangle and an angle that is opposite one of the given sides. An easy way to draw such a triangle is to:

(a) Draw a horizontal line, (this should be your initial ray), then draw the given angle counter-clockwise. The terminal ray should form the side of the triangle that is NOT opposite the given angle.

(b) Draw the side of the triangle that is opposite the given angle.

The horizontal line, the terminal ray and the side opposite the given angle should now form the required triangle.

Example 6 *To draw triangle ABC with $a = 9$ cm, $c = 16$ cm and $A = 30°$.*

In this triangle, the given angle is $A = 30°$ and the side opposite the given angle has length $a = 9$ cm. That side is BC. To draw the triangle, start by drawing a horizontal ray originating from a point you should label A, then draw an angle of 30° counterclockwise. Mark off a line segment AB of a reasonable length 16 and take it to be 16 cm, (see figure (a) below).. Next, draw a line segment BC joining B to the horizontal line. It should be shorter than AB because it is supposed to have length 9. The result is figure (b).

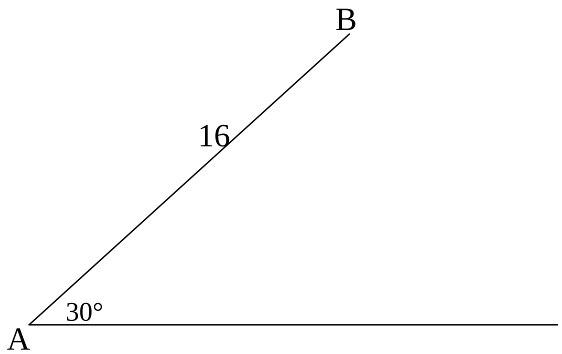


Figure (a)

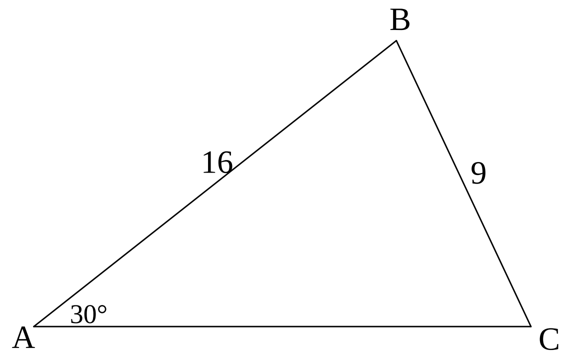
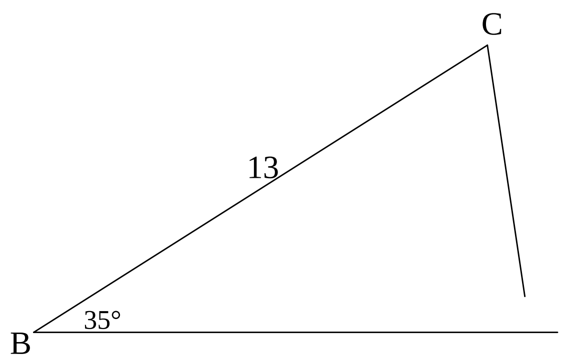


Figure (b)

There are three possibilities one may run into when solving a SSA problem.

1. The side opposite the given angle may be too short to form a triangle.

For an example, suppose you are asked to solve triangle ABC with $B = 35°$, $a = 13$ cm and $b = 7$ cm.



It so happens that the length b is too short to form a triangle. If you try to use the law of sines to find angle A, you get

$$\frac{\sin A}{13} = \frac{\sin 35°}{7} \qquad \text{OR} \qquad \sin A = \frac{12 \sin 35°}{7} = 1.065$$

which is impossible because the sine of an angle cannot exceed 1. **Conclusion**: There is no such a triangle.

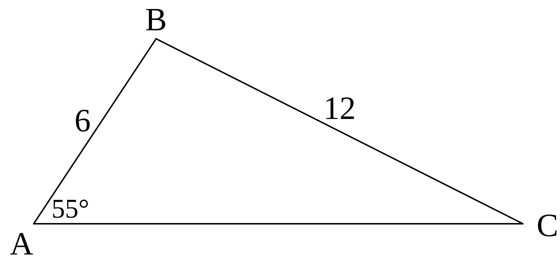
One way of checking if the side opposite the given angle is long enough to form a triangle is to compare it to the shortest distance from the horizontal line to the vertex above the horizontal line. In this case it is

$$13 \sin 35° = 7.46$$

which is longer than 7. Therefore b is not enough to form a rectangle.

2. The side opposite the given angle is bigger than the other given side.

   In this case only one triangle is possible. For example, suppose you are asked to solve the triangle ABC with $A = 55°$, $a = 12$ cm and $c = 6$ cm.



   In this case there is only one possible triangle. To solve it use the law of sines:

$$\frac{\sin 55°}{12} = \frac{\sin C}{6}$$

   Therefore $\sin C = \frac{6 \sin 55°}{12} = 0.5 \sin 55° = 0.4096$. This means that $C = 24.2°$. The second angle between 0 and 180 degrees whose sine is 0.4096 happens to be $(180 - 24.2)^{\circ} = 155.8°$. But C cannot be such an angle because B is 55° and the sum of the angles of a triangle is 180°. Therefore there is only one possible value of C and it is 24.2°. This means that $B = (180 - 55 - 24.2)^{\circ} = 100.8°$. Now we can find b, again using the law of sines:

$$\frac{b}{\sin 100.8°} = \frac{12}{\sin 55°}$$

   It follows that $b = \frac{12 \sin 100.8°}{\sin 55°} = 14.4$ (to 1 dec. pl.).

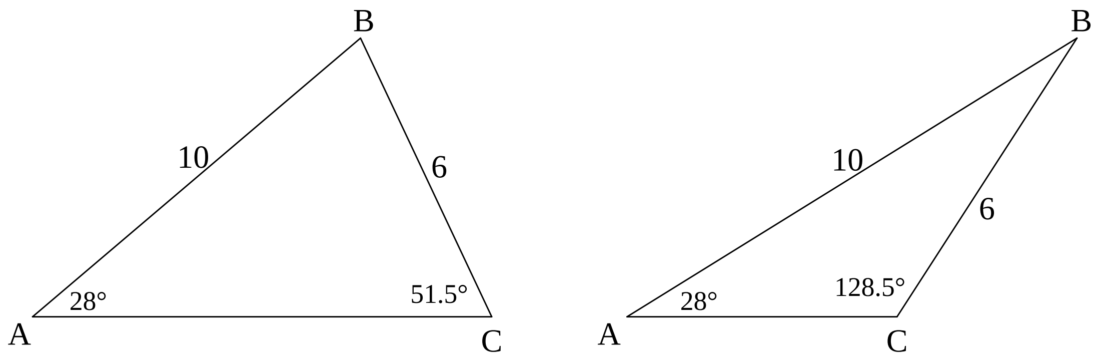
3. The side opposite the given angle is long enough to form a triangle and it is shorter than the other given side.

   In this case, two different triangles are possible. For example, suppose we are asked to solve triangle ABC with $A = 28°$, $c = 10$ cm and $a = 6$ cm. We use the Law of Sines. Since a and A are given the fraction $\frac{\sin A}{a}$ is known. We are also given c so that one of the numbers in the fraction $\frac{\sin C}{c}$ is known. We can, therefore, solve for $\sin C$ from

$$\frac{\sin C}{10} = \frac{\sin 28°}{6}$$

   The result is $\sin C = \frac{10 \sin 28°}{6} = 0.782\,45$. Therefore $C = \sin^{-1} 0.78245 = 51.5°$ or $C =$

$180° - 51.5° = 128.5°$, to one decimal place. The two possibilities are shown below.



In the triangle to the left, $B = 180° - 28° - 51.5° = 100.5°$. We use it to solve for b using the law of sines, (because we know$B$).

$$\frac{b}{\sin 100.5°} = \frac{6}{\sin 28°}$$

The result is $b = \dfrac{6 \sin 100.5°}{\sin 28°} = 12.5$ cm (to 1 decimal place), and the triangle is solved. In the other triangle, $B = 180° - 28° - 128.5° = 23.5°$. We can now solve for b using the law of sines.

$$\frac{b}{\sin 23.5°} = \frac{6}{\sin 28°}$$

The result is $b = \dfrac{6 \sin 23.5°}{\sin 28°} = 5.1$ cm (to 1 decimal place). This is also solved.

Exercise 7

1. *Show that there is no triangle ABC with* $C = 42°$, $c = 5.2$ *cm and* $a = 9.6$ *cm.*
2. *Solve the triangle ABC with* $B = 25°$, $b = 10$ *inches and* $c = 6$ *inches.*
3. *There are two triangles ABC with* $B = 35°$, $b = 11$ *cm and* $a = 16$ *cm. Draw them and solve each one.*
4. *You are required to solve a triangle ABC with* $B = 31°$, $c = 19$ *feet and* $b = 12$ *feet. There are two such possible triangles. Draw them then solve each one. Round off your values to the nearest whole number.*
5. *Solve the triangle ABC with* $a = 12$ *cm* $A = 47°$ *cm, and* $B = 75°$ *cm. Round off your values to the nearest whole number.*
6. *Solve the triangle ABC with* $B = 62°$, $a = 5$ *cm and* $c = 8$ *cm. Round off your values to the nearest whole number.*
7. *Solve the triangle ABC with* $a = 12$ *cm,* $b = 9$ *cm and* $c = 7$ *cm. Round off your values to the nearest whole number.*
8. *In each case, draw the triangle ABC then solve it. If there are two possible triangles, draw them and solve each.*
   (a) $B = 78°$, $C = 82°$, $b = 44$ *cm*
   (b) $B = 25°$, $c = 18$ *inches and* $b = 10$ *inches*
   (c) $b = 24$ *cm,* $c = 17$ *cm,* $C = 25°$.

Area of a Triangle

We need to introduce some standard terms used to calculate the area of a triangle.

Figure (a) shows a triangle ABC. We pick one of its three sides and call it the base of the triangle. For convenience, we have picked AB. We then drop a perpendicular from the vertex opposite the base, which is C, to the base itself. This is the line segment CD in Figure (b). The length of CD is called the height of the triangle.

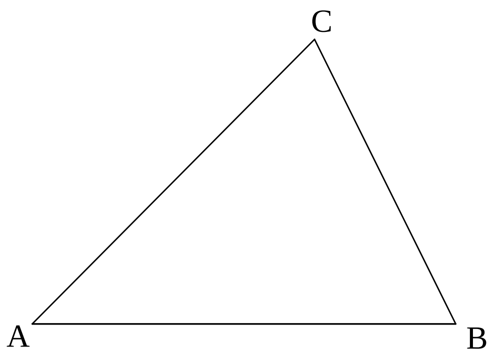


Figure (a)

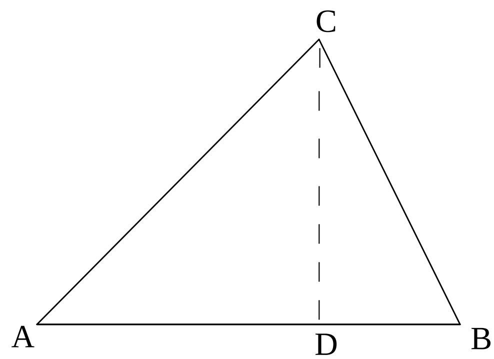
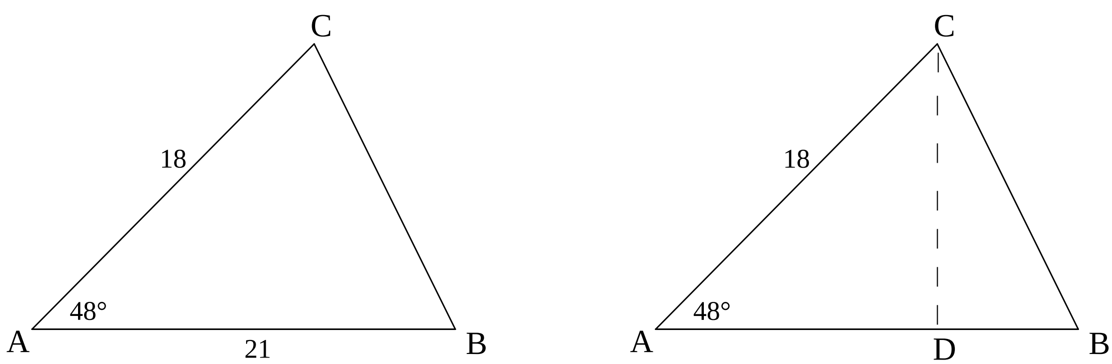


Figure (b)

Now we can give a formula for the area of the triangle ABC. It is

$$\text{Area of Triangle} = \frac{1}{2} \times (\text{Length of the Base of the Triangle}) \times (\text{Height of the Triangle})$$

Example 8 *In the left triangle ABC below,* $A = 48°$, $b = 18$ *cm and* $c = 21$ *cm. Thus the base of the triangle has length 21 cm.*



In the right triangle, we have added the height of the triangle. Since

$$\frac{\textit{Length of } DC}{\textit{Length of } AC} = \sin 48°$$

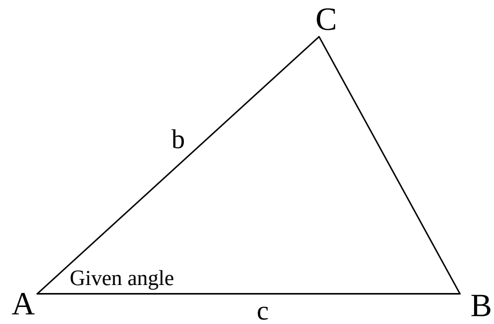
it follows that the height of the triangle is $(\textit{Length of } AC) \times (\sin 48°) = (18)\,(0.7431)$. *Therefore*

$$\textit{Area of Triangle } = \frac{1}{2} \times (21) \times (18)\,(0.7431) = 140.45 \textit{ sq. cm.}$$

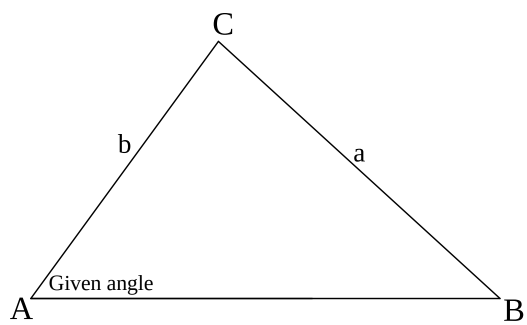
We rounded off the answer to 2 decimal places.

In general, if you are given two sides of a triangle and the angle between them then the area of the triangle is easy to calculate. Say you are given an angle A and the two sides b and c as shown below. Take AB as the base. The length of the base is c and the height of the triangle is $b \sin A$. Therefore the area of the triangle is

$$\frac{1}{2} bc \sin A$$



If you are given two sides and an angle opposite one of the given sides, you may proceed as follows: Say you are given angle A, and the two sides a and b as shown in the figure below.



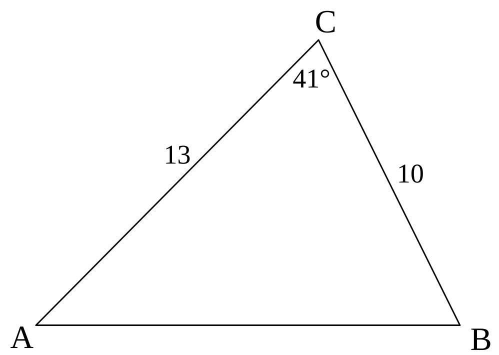
Calculate angle B. This enables you to calculate angle C. Now you have two sides and the angle between them, which you know how to handle.

If you are given two angles and a side, calculate one of the unknown sides. Now you know two sides and the angle between them, which you know how to handle.

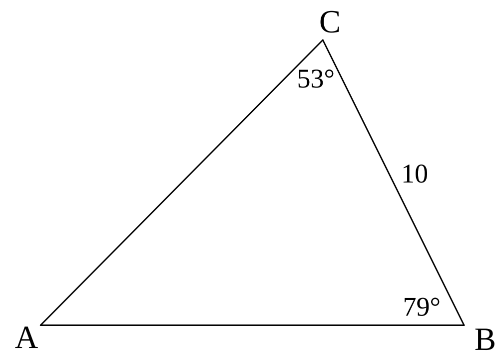
If you are given the lengths of the three sides of a triangle ABC and no angle, you may calculate its area using Heron's formula. Say the sides have lengths a, b, and c. You have to determine the number $s = \frac{1}{2}(a + b + c)$. Then according to Heron's theorem, the area of the triangle is

$$\sqrt{s(s-a)(s-b)(s-c)}$$

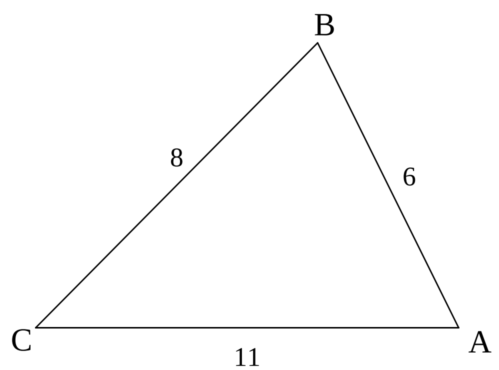
Exercise 9 *Calculate the area of each triangle*



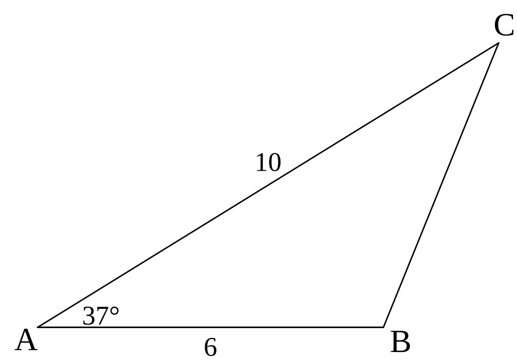
(i)



(ii)

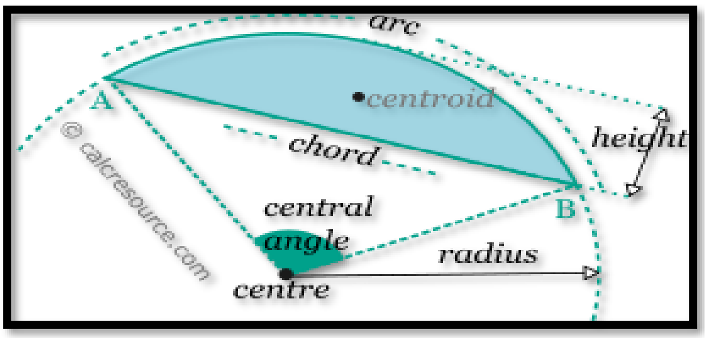


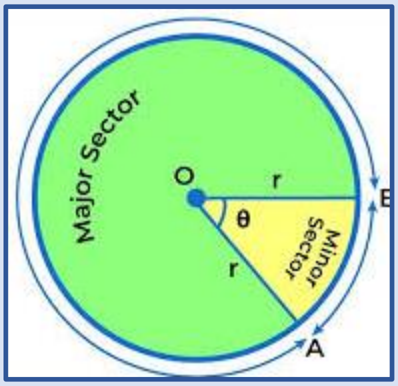
(iii)



(iv)

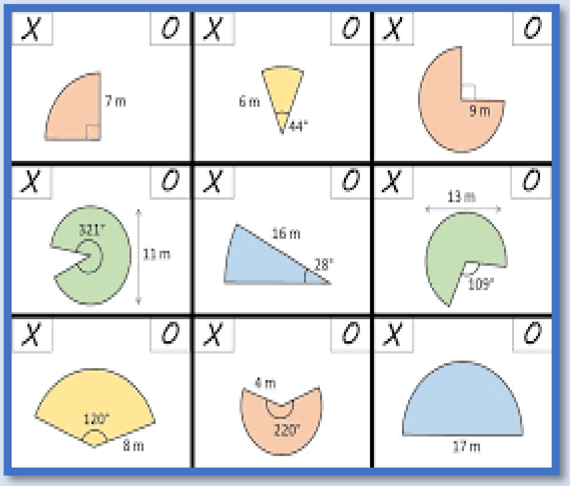
Lecture Nine





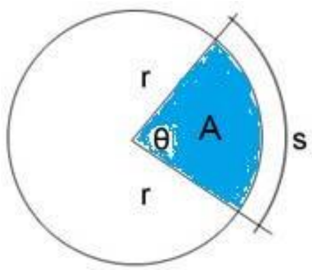
Behavioral objectives

1. Convert angles between degrees and radians.
2. Calculate the area of a circular sector given the angle and radius.
3. Calculate arc length of a sector using appropriate formulas.
4. Apply formulas in engineering contexts involving circular shapes.



Area and Arc Length of a Sector

Calculating the Area of a Sector:



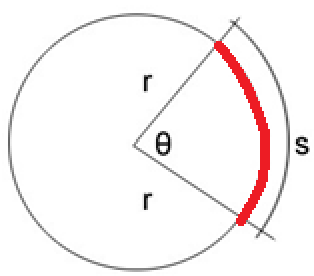
Area, A, of a sector, with radius. r, and subtended angle, θ, in radians is given by:

$$A = \frac{1}{2}\theta r^2$$

Note: if θ is given in degrees, it must be converted into radians first. The following formula can be used:

$$A = \frac{1}{2}\theta^0\left(\frac{\pi}{180^0}\right)r^2$$

Calculating the Arc Length of a Sector:



Arc length, s, of a sector with radius, r, and subtended angle, θ, in radians is given by:

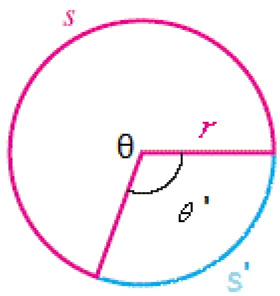
$$S = \theta r$$

Note: if θ is given in degrees, it must be converted into radians first. The following formula can be used:

$$S = \theta^0\left(\frac{\pi}{180^0}\right)r$$

Area and Arc Length of a Sector

Example 3.4.
Find the lengths of the arcs s and s' in the figure if r = 4 and θ'= 60°.



Solution:

To find the arc length, s, first we have to find the angle θ that subtends the arc s,

$\theta + \theta' = 360^o$

$\theta = 360^o - \theta' = 360^o – 60^o = 300^o$

Now, we can apply the formula for finding the length of an arc if the angle is given in degrees.

<u>To find the length of s:</u>

$s = \theta^0 (\frac{\pi}{180^0}) r$

$= 300(\frac{\pi}{180^0})(4)$

$= \frac{20\pi}{3}$

<u>To find the length of s':</u>

$s' = \theta'^0 (\frac{\pi}{180^0})(r)$

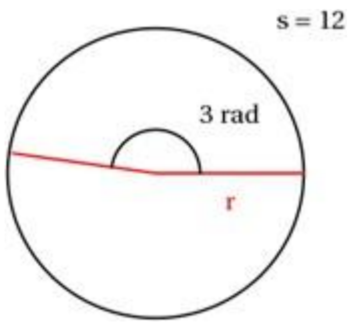
$= 60(\frac{\pi}{180^0})(4)$

$= \frac{4\pi}{3}$

Area and Arc Length of a Sector

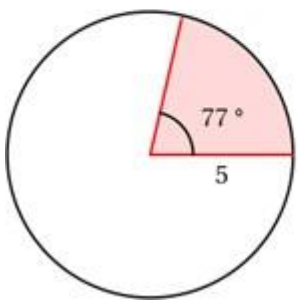
Practice Questions:

1. Find the radius r of the circle in the figure with arc length s.

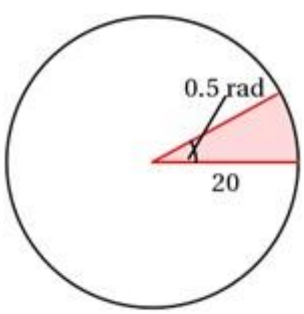


2. Find the length of an arc that subtends a central angle of 3 rad in a circle of radius 8 mi.

3. Find the area of the sectors in the following diagrams:



a)



b)

4. Find the area of a sector with central angle 1 rad in a circle of radius 14 m.

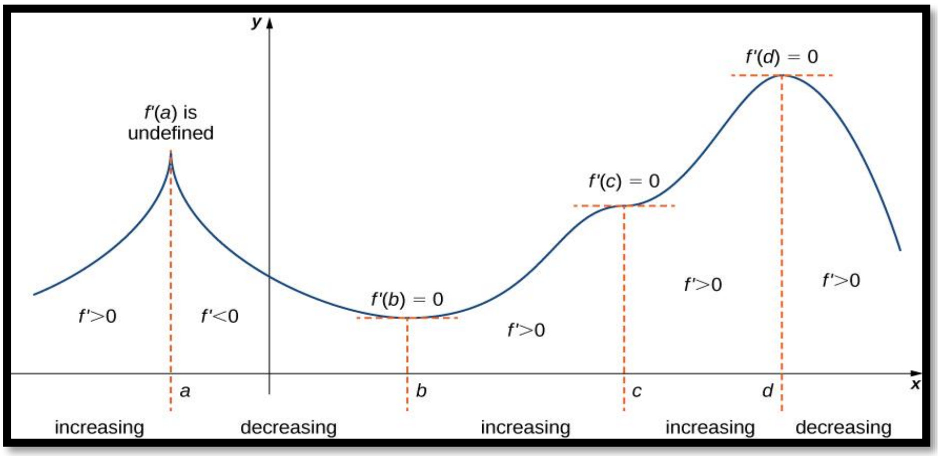
5. The area of a sector of a circle with a central angle of 4 rad is 8 m^2. Find the radius of the circle.

Area and Arc Length of a Sector

<u>Answers:</u>

1) 4
2) 24 mi
3) a) 5.35π
   b) 100
4) 98 m^2
5) 2 m

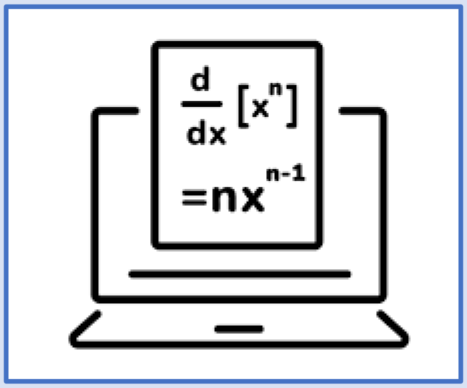
Lecture Ten



y
f′(d) = 0
f′(a) is undefined
f′(c) = 0
f′(b) = 0
f′>0
f′<0
f′>0
f′>0
f′>0
a
b
c
d
x
increasing
decreasing
increasing
increasing
decreasing

Behavioral objectives

1. Understand the concept of the derivative as a rate of change.
2. Apply rules of differentiation to algebraic and polynomial functions.
3. Analyze and sketch graphs using first derivatives (monotonicity and critical points).
4. Determine extreme values of functions using calculus methods.



Implicit Differentiation

In the function that is difficult to separate x from y, for example:

$$x^2 + y^2 - 25 = 0, \quad y^2 - x = 0, \quad or \quad x^3 + y^3 - 9xy = 0$$

These equations define an *implicit* relation between the variables x and y, meaning that a value of x determines one or more values of y, even though we do not have a simple formula for the y-values.

Implicit Differentiation

1. Differentiate both sides of the equation with respect to x, treating y as a differentiable function of x.
2. Collect the terms with dy/dx on one side of the equation and solve for dy/dx.

Example 1: find dy/dx for the equation $y^2 = x^2 + \sin xy$

Solution:

$$\frac{d}{dx}(y^2) = \frac{d}{dx}(x^2) + \frac{d}{dx}(\sin xy)$$

$$2y\frac{dy}{dx} = 2x + (\cos xy)\left(x\frac{dy}{dx} + y\right)$$

$$2y\frac{dy}{dx} - (\cos xy)\left(x\frac{dy}{dx} + y\right) = 2x$$

$$2y\frac{dy}{dx} - (\cos xy)\left(x\frac{dy}{dx}\right) = 2x + (\cos xy)y$$

$$(2y - x(\cos xy))\frac{dy}{dx} = 2x + y(\cos xy)$$

$$\frac{dy}{dx} = \frac{2x + y(\cos xy)}{2y - x(\cos xy)}$$

Example 2: find d^2y/dx^2 for the equation $2x^3 - 3y^2 = 8$

Solution:

$$\frac{d}{dx}(2x^3) - \frac{d}{dx}(3y^2) = \frac{d}{dx}(8)$$

$$6x^2 - 6yy' = 0$$

$$x^2 - yy' = 0$$

$$y' = \frac{x^2}{y}, \quad y \neq 0$$

$$y'' = \frac{2xy - x^2y'}{y^2} = \frac{2x}{y} - \frac{x^2}{y^2}.y'$$

Substitute $y' = \frac{x^2}{y}$ in the y'':

$$y'' = \frac{2x}{y} - \frac{x^2}{y^2} \cdot (\frac{x^2}{y}) = \frac{2x}{y} - \frac{x^4}{y^3}, \quad y \neq 0$$

Example 3: Find the derivative of the following function:

$$x^2y^2 = x^2 + y^2$$

$$x^2.2yy' + 2xy^2 = 2x + 2yy' \quad \div 2$$

$$x^2.yy' - yy' = x - xy^2$$

$$(x^2.y - y)y' = x - xy^2$$

$$y' = \frac{(1 - y^2)x}{(x^2 - 1)y}$$

Lecture 10

Extreme Values of Functions

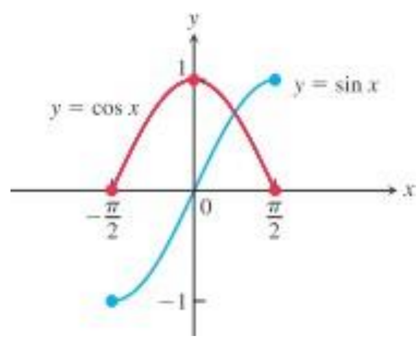
Definitions: Let f be a function with domain D. Then f has an **absolute maximum** value on D at a point c if:

$$f(x) \leq f(c) \quad for\ all\ x\ in\ D$$

and an **absolute minimum** value on D at c if:

$$f(x) \geq f(c) \quad for\ all\ x\ in\ D$$

Absolute maximum and minimum values are called extreme values of the function f. Absolute maxima or minima are also referred to as global maxima or minima.



Finding Extreme Values

If f has a local maximum or minimum value at an interior point c of its domain, and if f' is defined at c, then:

$$f'(c) = 0.$$

Critical Point

An interior point of the domain of a function f where f' is zero or undefined is a critical point of f.

How to find the absolute extreme values of a function?

1. Find all critical points of f on the interval.
2. Evaluate f at all **critical points** and **endpoints**.
3. Take the largest and smallest of these values.

Example 4: Find the absolute maximum and minimum values of $f(x) = x^2$ over [-2, 1]

Solution: $f'(x) = 2x = 0,$ critical point $at\ x = 0$

$$f'(0) = 0$$

$$f(-2) = 4$$

$$f(1) = 1$$

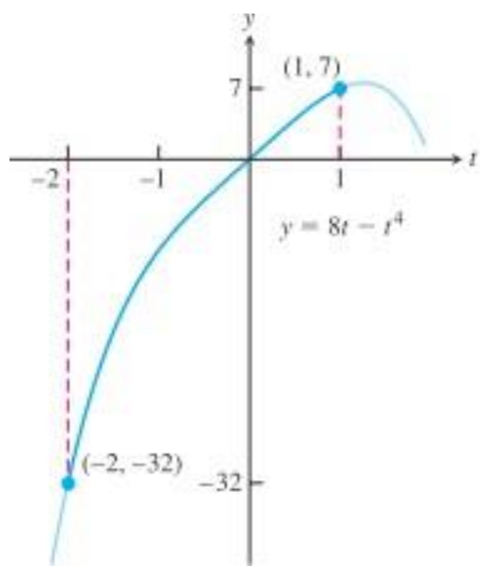
The function has an absolute maximum value of 4 at x= -2, and an absolute minimum value of 0 at x=0.

Example 5: Find the absolute extreme values of $g(t) = 8t - t^4$ over [-2, 1]

Solution: $g'(t) = 8 - 4t^3 = 0,$ $t = 2^{1/3} > 1.$

The point is not in the given domain. Therefore, the function's absolute extreme values occur at the endpoints.

The absolute **minimum**$g(-2) = -32$, and the absolute **maximum** $g(1) = 7$.



Monotonic Functions and the First Derivative Test

In sketching the graph of a differentiable function, it is useful to know where it increases (rises from left to right) and where it decreases (falls from left to right) over an interval.

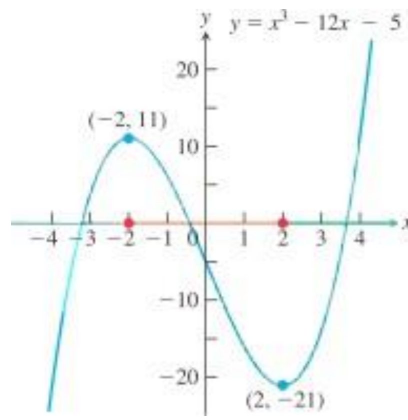
Example 6: Find the critical points of $f(x) = x^3 - 12x - 5$ and identify the open intervals on which f is increasing and on which f is decreasing.

Solution: The first derivative of the function is:

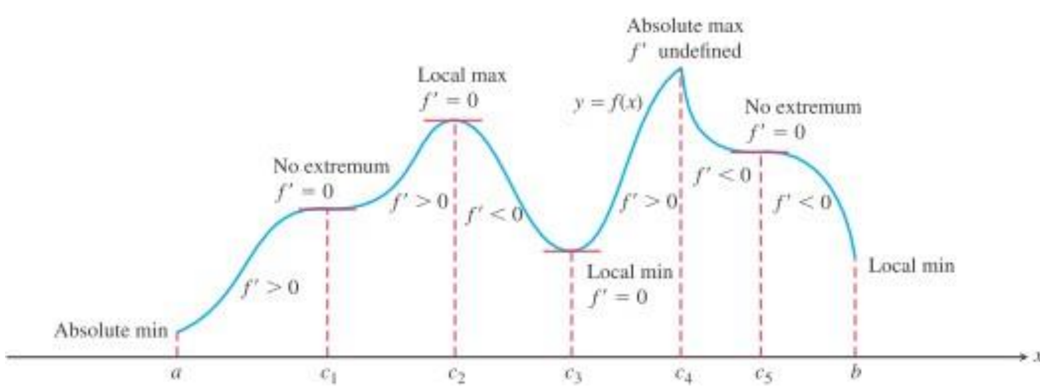
$$f'(x) = 3x^2 - 12 = 3(x^2 - 4)$$
$$= 3(x+2)(x-2) = 0$$

By equaling the first derivative to zero, $x = 2$, $x = -2$. These critical points subdivide the domain of f into intervals (-∞, -2), (-2, 2), and (2, ∞) on which f' is either positive or negative. We determine the sign of f' by evaluating f' at a convenient point in each subinterval.

Intervals	***-∞ <x< -2***	***-2 <x< 2***	***2 <x< ∞***
f′	***f′*(-3)=15**	***f′*(0)= -12**	***f′*(3)=15**
Sign of *f′*	+	**-**	+
Behavior of *f*	increasing	decreasing	increasing



First Derivative Test for Local Extremes



1. if f' changes from negative to positive at c, then f has a local **minimum** at c;
2. if f' changes from positive to negative at c, then f has a local **maximum** at c;
3. if f' does not change sign at c (that is, f' is positive on both sides of c or negative on both sides), then f has no local extreme value at c.

Example 7: Find the critical points of

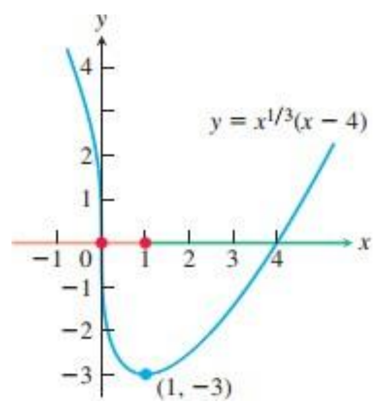
$$f(x) = x^{\frac{1}{3}}(x-4) = x^{4/3} - 4x^{1/3}$$

Solution:

$$f'(x) = \frac{4}{3}x^{1/3} - \frac{4}{3}x^{-2/3} = \frac{4}{3}x^{-\frac{2}{3}}(x-1) = \frac{4(x-1)}{3x^{2/3}} = 0$$

The derivative is zero at x=1, and undefined at x=0. There are no endpoints in the domain. Therefore, the critical points are x=0 and x=1.

Interval	x<1	0<x<1	x>1
Sign of f'	-	-	+
Behavior of f	decreasing	decreasing	increasing



The First Derivative Test for Local Extreme tells us that f does not have an extreme value at x = 0 (f' does not change sign) and that f has a local minimum at x = 1 (f' changes from negative to positive). The value of the local minimum is

$$f(1) = x^{\frac{1}{3}}(x-4) = 1^{4/3} - 4(1)^{1/3} = -3$$

Lecture 10

Homework

1. Find the derivative (dy/dx) of the following functions:

   a) $xy = \cot xy$

   b) $x\cos(2x+3y) = y\sin x$

   c) $\cos y + \cot x = xy$

   d) $(3xy+7)^2 = 6y$

   Find d^2y/dx^2 for the following equations:

   e) $xy + y^2 = 1$

   f) $x^{2/3} + y^{2/3} = 1$

   g) $x^3 + y^4 = 16$

2. Find the extreme values of each function on the given interval. Then plot the function and identify the points where the absolute extreme occur:

   a) $f(x) = \frac{2x}{3} - 5, \ -2 \le x \le 3$

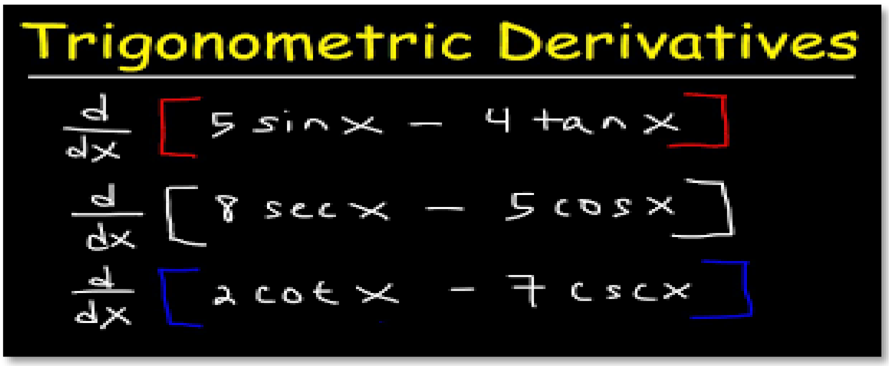
   b) $f(x) = -x^{-2}, \ -0.5 \le x \le 2$

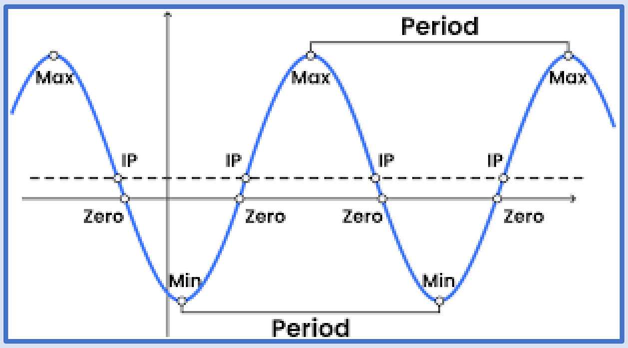
   c) $f(x) = -3x^{2/3}, \ -1 \le x \le 1$

   d) $f(x) = \sin x, -\pi/2 \le x \le 5\pi/6$

   e) $f(x) = \sec x, -\pi/3 \le x \le \pi/6$

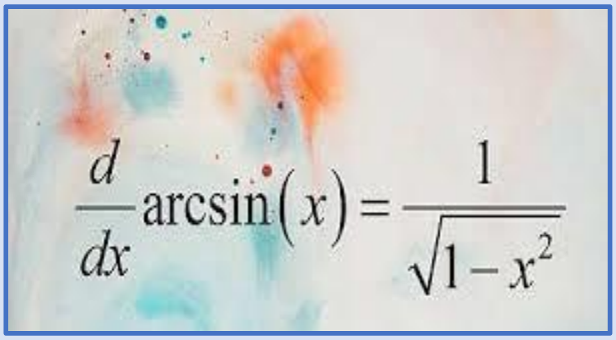
Lecture Eleven





Behavioral objectives

1. Differentiate the basic trigonometric functions (sin, cos, tan, etc.).
2. Apply trigonometric differentiation to solve problems in physics and engineering.
3. Understand the behavior of trigonometric curves using derivatives.
4. Solve related problems involving velocity and slopes of curves.



Derivatives of Trigonometric Functions

The derivative of the sine function is the cosine function:

$$\frac{d}{dx}(\sin x) = \cos x.$$

The derivative of the cosine function is the negative of the sine function:

$$\frac{d}{dx}(\cos x) = -\sin x.$$

The derivatives of the other trigonometric functions:

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

Many phenomena of nature are approximately periodic (electromagnetic fields, heart rhythms, tides, weather). The derivatives of sines and cosines play a key role in describing periodic changes. This section shows how to differentiate the six basic trigonometric functions.

EXAMPLE 1 Derivative of the Sine Function

To calculate the derivative of $f(x) = \sin x$, for x measured in radians, we combine the limits with the angle sum identity for the sine function:

If $f(x) = \sin x$, then $\sin(x + h) = \sin x \cos h + \cos x \sin h$.

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{\sin(x + h) - \sin x}{h} \qquad \text{Derivative definition}$$

$$= \lim_{h \to 0} \frac{(\sin x \cos h + \cos x \sin h) - \sin x}{h} = \lim_{h \to 0} \frac{\sin x(\cos h - 1) + \cos x \sin h}{h}$$

$$= \lim_{h \to 0} \left(\sin x \cdot \frac{\cos h - 1}{h}\right) + \lim_{h \to 0} \left(\cos x \cdot \frac{\sin h}{h}\right)$$

$$= \sin x \cdot \underbrace{\lim_{h \to 0} \frac{\cos h - 1}{h}}_{\text{limit 0}} + \cos x \cdot \underbrace{\lim_{h \to 0} \frac{\sin h}{h}}_{\text{limit 1}} = \sin x \cdot 0 + \cos x \cdot 1 = \cos x.$$

EXAMPLE 2 Derivative of the Cosine Function

With the help of the angle sum formula for the cosine function,

$$\cos(x + h) = \cos x \cos h - \sin x \sin h,$$

we can compute the limit of the difference quotient:

$$\frac{d}{dx}(\cos x) = \lim_{h \to 0} \frac{\cos(x + h) - \cos x}{h} \qquad \text{Derivative definition}$$

$$= \lim_{h \to 0} \frac{(\cos x \cos h - \sin x \sin h) - \cos x}{h} \qquad \text{Cosine angle sum identity}$$

$$= \lim_{h \to 0} \frac{\cos x(\cos h - 1) - \sin x \sin h}{h}$$

$$= \lim_{h \to 0} \cos x \cdot \frac{\cos h - 1}{h} - \lim_{h \to 0} \sin x \cdot \frac{\sin h}{h}$$

$$= \cos x \cdot \lim_{h \to 0} \frac{\cos h - 1}{h} - \sin x \cdot \lim_{h \to 0} \frac{\sin h}{h}$$

$$= \cos x \cdot 0 - \sin x \cdot 1$$

$$= -\sin x.$$

EXAMPLE 3 We find derivatives of the function involving differences, products, and quotients.

(a) $y = x^2 - \sin x$:

$$\frac{dy}{dx} = 2x - \frac{d}{dx}(\sin x) \qquad \text{Difference Rule}$$

$$= 2x - \cos x$$

(b) $y = \dfrac{\sin x}{x}$:

$$\frac{dy}{dx} = \frac{x \cdot \frac{d}{dx}(\sin x) - \sin x \cdot 1}{x^2} \qquad \text{Quotient Rule}$$

$$= \frac{x \cos x - \sin x}{x^2}$$

(c) $y = \sin x \cos x$:

$$\frac{dy}{dx} = \sin x \frac{d}{dx}(\cos x) + \cos x \frac{d}{dx}(\sin x) \qquad \text{Product Rule}$$

$$= \sin x(-\sin x) + \cos x(\cos x)$$

$$= \cos^2 x - \sin^2 x$$

■

(e) $y = \dfrac{\cos x}{1 - \sin x}$:

$$\frac{dy}{dx} = \frac{(1 - \sin x)\frac{d}{dx}(\cos x) - \cos x \frac{d}{dx}(1 - \sin x)}{(1 - \sin x)^2} \qquad \text{Quotient Rule}$$

$$= \frac{(1 - \sin x)(-\sin x) - \cos x(0 - \cos x)}{(1 - \sin x)^2}$$

$$= \frac{1 - \sin x}{(1 - \sin x)^2} \qquad \sin^2 x + \cos^2 x = 1$$

$$= \frac{1}{1 - \sin x}$$

■

EXAMPLE 4 Find $d(\tan x)/dx$.

Solution We use the Derivative Quotient Rule to calculate the derivative:

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x \frac{d}{dx}(\sin x) - \sin x \frac{d}{dx}(\cos x)}{\cos^2 x} \qquad \text{Quotient Rule}$$

$$= \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x} = \sec^2 x.$$

■

EXAMPLE 5 Find y'' if $y = \sec x$.

Solution Finding the second derivative involves a combination of trigonometric derivatives.

$$y = \sec x$$

$$y' = \sec x \tan x \qquad \text{Derivative rule for secant function}$$

$$y'' = \frac{d}{dx}(\sec x \tan x)$$

$$= \sec x \frac{d}{dx}(\tan x) + \tan x \frac{d}{dx}(\sec x) \qquad \text{Derivative Product Rule}$$

$$= \sec x(\sec^2 x) + \tan x(\sec x \tan x) \qquad \text{Derivative rules}$$

$$= \sec^3 x + \sec x \tan^2 x$$

■

Exercises 3.5

Derivatives

In Exercises 1–18, find dy/dx.

1. $y = -10x + 3\cos x$
2. $y = \frac{3}{x} + 5\sin x$
3. $y = x^2 \cos x$
4. $y = \sqrt{x}\sec x + 3$
5. $y = \csc x - 4\sqrt{x} + 7$
6. $y = x^2 \cot x - \frac{1}{x^2}$
7. $f(x) = \sin x \tan x$
8. $g(x) = \csc x \cot x$
9. $y = (\sec x + \tan x)(\sec x - \tan x)$
10. $y = (\sin x + \cos x)\sec x$
11. $y = \frac{\cot x}{1 + \cot x}$
12. $y = \frac{\cos x}{1 + \sin x}$
13. $y = \frac{4}{\cos x} + \frac{1}{\tan x}$
14. $y = \frac{\cos x}{x} + \frac{x}{\cos x}$
15. $y = x^2 \sin x + 2x\cos x - 2\sin x$
16. $y = x^2 \cos x - 2x\sin x - 2\cos x$
17. $f(x) = x^3 \sin x \cos x$
18. $g(x) = (2 - x)\tan^2 x$

In Exercises 19–22, find ds/dt.

19. $s = \tan t$
20. $s = t^2 - \sec t$
21. $s = \frac{1 + \csc t}{1 - \csc t}$
22. $s = \frac{\sin t}{1 - \cos t}$

In Exercises 23–26, find $dr/d\theta$.

23. $r = 4 - \theta^2 \sin\theta$
24. $r = \theta\sin\theta + \cos\theta$
25. $r = \sec\theta\csc\theta$
26. $r = (1 + \sec\theta)\sin\theta$

In Exercises 27–32, find dp/dq.

27. $p = 5 + \frac{1}{\cot q}$
28. $p = (1 + \csc q)\cos q$
29. $p = \frac{\sin q + \cos q}{\cos q}$
30. $p = \frac{\tan q}{1 + \tan q}$
31. $p = \frac{q\sin q}{q^2 - 1}$
32. $p = \frac{3q + \tan q}{q\sec q}$
33. Find y'' if
    a. $y = \csc x$.
    b. $y = \sec x$.
34. Find $y^{(4)} = d^4y/dx^4$ if
    a. $y = -2\sin x$.
    b. $y = 9\cos x$.

Tangent Lines

In Exercises 35–38, graph the curves over the given intervals, together with their tangents at the given values of x. Label each curve and tangent with its equation.

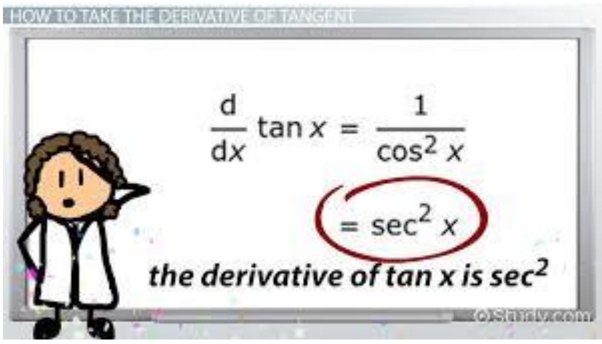
35. $y = \sin x, \quad -3\pi/2 \le x \le 2\pi$
    $x = -\pi, 0, 3\pi/2$
36. $y = \tan x, \quad -\pi/2 < x < \pi/2$
    $x = -\pi/3, 0, \pi/3$
37. $y = \sec x, \quad -\pi/2 < x < \pi/2$
    $x = -\pi/3, \pi/4$
38. $y = 1 + \cos x, \quad -3\pi/2 \le x \le 2\pi$
    $x = -\pi/3, 3\pi/2$

Trigonometric Limits

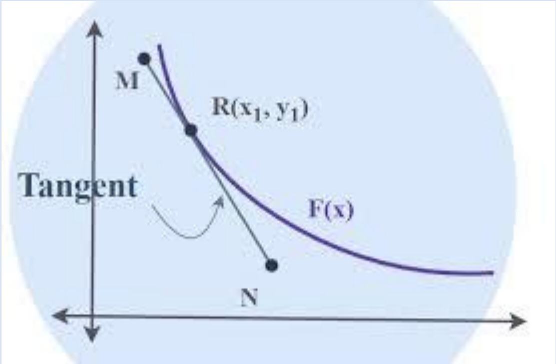
Find the limits in Exercises 47–54.

47. $\lim_{x\to 2} \sin\left(\frac{1}{x} - \frac{1}{2}\right)$
48. $\lim_{x\to -\pi/6} \sqrt{1 + \cos(\pi\csc x)}$
49. $\lim_{\theta\to\pi/6} \frac{\sin\theta - \frac{1}{2}}{\theta - \frac{\pi}{6}}$
50. $\lim_{\theta\to\pi/4} \frac{\tan\theta - 1}{\theta - \frac{\pi}{4}}$
51. $\lim_{x\to 0} \sec\left[e^x + \pi\tan\left(\frac{\pi}{4\sec x}\right) - 1\right]$
52. $\lim_{x\to 0} \sin\left(\frac{\pi + \tan x}{\tan x - 2\sec x}\right)$
53. $\lim_{t\to 0} \tan\left(1 - \frac{\sin t}{t}\right)$
54. $\lim_{\theta\to 0} \cos\left(\frac{\pi\theta}{\sin\theta}\right)$

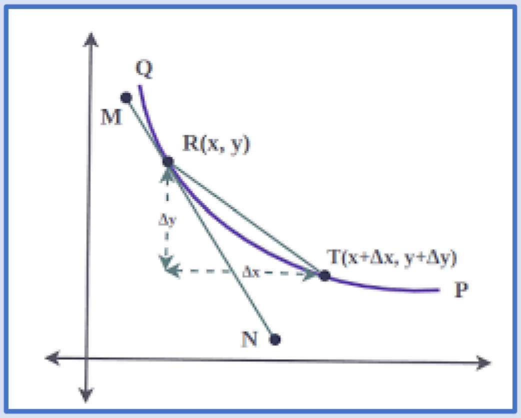
Lecture Twelve



Behavioral objectives



1. Find the slope of a tangent to a curve at a given point.
2. Use limits to calculate derivatives from first principles.
3. Apply derivative knowledge to motion, velocity, and optimization problems.
4. Interpret the geometric meaning of derivatives on graphs.



L12

Differentiation

The problem of finding the tangent line to a curve and the problem of finding the velocity of an object both involve finding the same type of limit, as we saw in chapter two. This special type of limit is called a *derivative* and we will see that it can be interpreted as a rate of change in any of the sciences or engineering.

3.1 Tangents and the Derivative at a Point

To find a tangent to an arbitrary curve $y = f(x)$ at a point $P(x_o, f(x_o))$, we calculate the slope of the secant through P and a nearby point $Q(x_o + h, f(x_o + h))$. We then investigate the limit of the slope as $h \to 0$ (Figure 1). If the limit exists, we call it the slope of the curve at P and define the tangent at P to be the line through P having this slope.

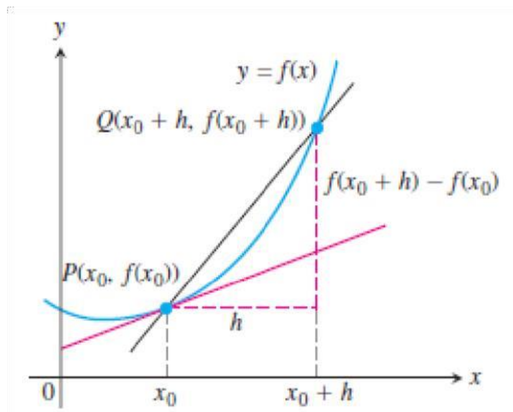


Figure 1

DEFINITIONS The **slope of the curve** $y = f(x)$ at the point $P(x_0, f(x_0))$ is the number

$$m = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \quad \text{(provided the limit exists).}$$

The **tangent line** to the curve at P is the line through P with this slope.

Example 1

(a) Find the slope of the curve $y = 1/x$ at any point $x = a \neq 0$. What is the slope at the point $x = -1$?

(b) Where does the slope equal $-\frac{1}{4}$?

(c) What happens to the tangent to the curve at the point $(a, 1/a)$ as a changes? **Solution**:

(a) Here $f(x) = 1/x$. The slope at $(a, 1/a)$ is

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{\frac{1}{a+h} - \frac{1}{a}}{h} = \lim_{h \to 0} \frac{1}{h} \frac{a - (a+h)}{a(a+h)}$$

$$= \lim_{h \to 0} \frac{-h}{ha(a+h)} = \lim_{h \to 0} \frac{-1}{a(a+h)} = -\frac{1}{a^2}.$$

Notice how we had to keep writing "$\lim_{h \to 0}$" before each fraction until the stage where we could evaluate the limit by substituting $h = 0$. The number a may be positive or negative, but not 0. When $a = -1$, the slope is $-1/(-1)^2 = -1$ (Figure 2).

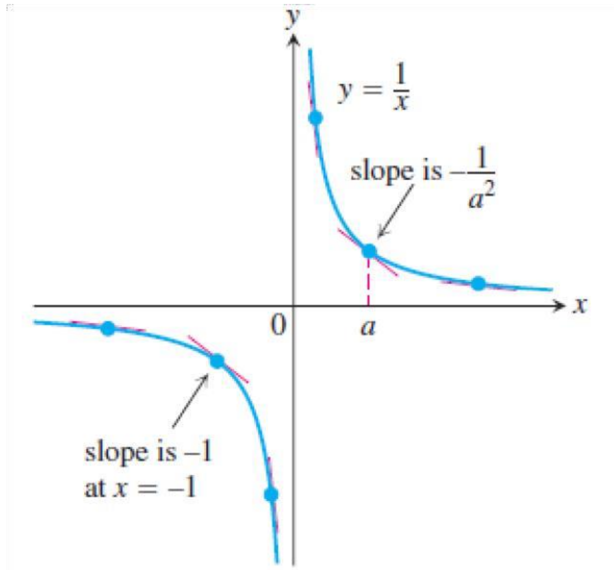


Figure 2

(b) The slope of $y = 1/x$ at the point where $x = a$ is $-1/a^2$. It will be provided that

$$-1/a^2 = -1/4$$

This equation is equivalent to $a^2 = 4$, so $a = 2$ or $a = -2$. The curve has slope -1/4 at the two points (2, 1/2) and (-2, -1/2) (Figure 3).

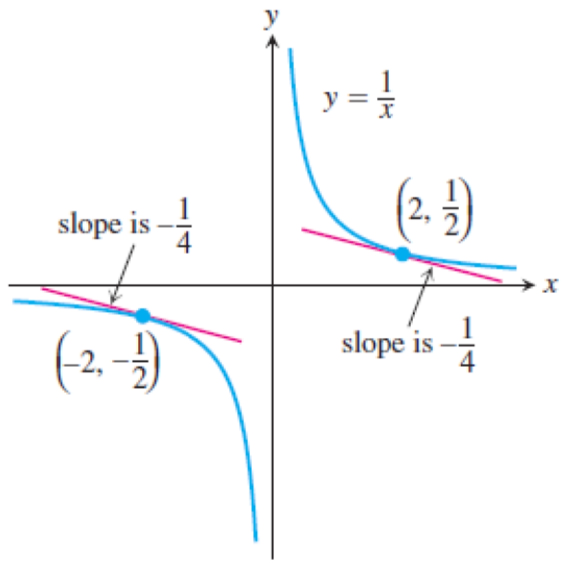


Figure 3

(c) The slope $-1/a^2$ is always negative if $a \neq 0$. As $a \to 0^+$, the slope approaches $-\infty$ and the tangent becomes increasingly steep (Figure 3). We see this situation again as $a \to 0^-$. As a moves away from the origin in either direction, the slope approaches 0 and the tangent levels off to become horizontal.

3.2 Rates of Change: Derivative at a Point

The expression

$$\frac{f(x_0 + h) - f(x_0)}{h}, \quad h \neq 0$$

is called the **difference quotient of f at x_0 with increment h**. If the difference quotient has a limit as h approaches zero, that limit is given a special name and notation.

DEFINITION The **derivative of a function f at a point x_0**, denoted $f'(x_0)$, is

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

provided this limit exists.

3.3 Summary
All of these ideas refer to the same limit.

The following are all interpretations for the limit of the difference quotient,

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

1. The slope of the graph of $y = f(x)$ at $x = x_0$
2. The slope of the tangent to the curve $y = f(x)$ at $x = x_0$
3. The rate of change of $f(x)$ with respect to x at $x = x_0$
4. The derivative $f'(x_0)$ at a point

3.4 The Derivative as a Function

In the last section we defined the derivative of $y = f(x)$ at the point $x = x_0$ to be the limit

$$f(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

We now investigate the derivative as a *function* derived from f by considering the limit at each point x in the domain of f.

DEFINITION The **derivative** of the function $f(x)$ with respect to the variable x is the function f' whose value at x is

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h},$$

provided the limit exists.

The process of calculating a derivative is called **differentiation**. To emphasize the idea that differentiation is an operation performed on a function $y = f(x)$, we use the notation

$$\frac{d}{dx}f(x)$$

There are many ways to denote the derivative of a function *y = f(x),* where the independent variable is *x* and the dependent variable is y. Some common alternative notations for the derivative are:

$$f(x) = y = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = D(f)(x) = D_x f(x)$$

Example 2: by using the definition of the derivative, find *dy/dx* of the function $y = 5x^3 + 8x^2 - 3x + 4$ **Solution:**

$f(x) = 5x^3 + 8x^2 - 3x + 4$ $f(x + \Delta x) = 5(x + \Delta x)^3 + 8(x + \Delta x)^2 - 3(x + \Delta x) + 4$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{x + \quad - \quad x}{\Delta x} \qquad f(\qquad h)$$

$$\lim_{\Delta x \to 0} \frac{5(x + \Delta x)^3 + 8(x + \Delta x)^2 - 3(x + \Delta x) + 4 - 5x^3 - 8x^2 + 3x - 4}{\Delta x}$$

( )

$$\lim_{\Delta x \to 0} \frac{5(x^3 + 3x^2\,\Delta x + 3x\Delta x^2 + \Delta x^3) + 8(x^2 + 2x\Delta x + \Delta x^2) - 3x - 3\Delta x + 4 - 5x^3 - 8x^2 + 3x - 4}{\Delta x}$$

$$\lim_{\Delta x \to 0} \frac{5x^3 + 15x^2\,\Delta x + 15x\Delta x^2 + 5\Delta x^3 + 8x^2 + 16x\Delta x + 8\Delta x^2 - 3x - 3\Delta x - 5x^3 - 8x^2 + 3x}{\Delta x}$$

$$\lim_{\Delta x \to 0} \frac{15x^2\,\Delta x + 15x\Delta x^2 + 5\Delta x^3 + 16x\Delta x + 8\Delta x^2 - 3\Delta x}{\Delta x}$$

$$\lim_{\Delta x \to 0} 15x^2 + 15x\Delta x + 5\Delta x^2 + 16x + 8\Delta x - 3$$

$= 15\,x^2 + 16\,x - 3$

3.5 Differentiation Rules

> **Derivative of a Constant Function**
>
> If f has the constant value $f(x) = c$, then
>
> $$\frac{df}{dx} = \frac{d}{dx}(c) = 0.$$

Example 3: $\frac{d}{dx}5 = 0$

> **Power Rule (General Version)**
>
> If n is any real number, then
>
> $$\frac{d}{dx}x^n = nx^{n-1},$$
>
> for all x where the powers x^n and x^{n-1} are defined.

Example 4: Differentiate the following equations:

(a) x^3, (b) $x^{2/3}$, (c) $\sqrt{\ }$, (d) $\frac{1}{x^4}$, (e) $x^{-4/3}$, (f) $\sqrt{x^{2+\pi}}$

Solution:

(a) $\frac{d}{dx}(x^3) = 3x^{3-1} = 3x^2$ (b) $\frac{d}{dx}(x^{2/3}) = \frac{2}{3}x^{(2/3)-1} = \frac{2}{3}x^{-1/3}$

(c) $\frac{d}{dx}\left(x^{\sqrt{2}}\right) = \sqrt{2}x^{\sqrt{2}-1}$ (d) $\frac{d}{dx}\left(\frac{1}{x^4}\right) = \frac{d}{dx}(x^{-4}) = -4x^{-4-1} = -4x^{-5} = -\frac{4}{x^5}$

(e) $\frac{d}{dx}(x^{-4/3}) = -\frac{4}{3}x^{-(4/3)-1} = -\frac{4}{3}x^{-7/3}$

(f) $\frac{d}{dx}\left(\sqrt{x^{2+\pi}}\right) = \frac{d}{dx}\left(x^{1+(\pi/2)}\right) = \left(1 + \frac{\pi}{2}\right)x^{1+(\pi/2)-1} = \frac{1}{2}(2 + \pi)\sqrt{x^{\pi}}$ ■

Derivative Constant Multiple Rule

If u is a differentiable function of x, and c is a constant, then

$$\frac{d}{dx}(cu) = c\frac{du}{dx}.$$

Example 5: Differentiate the equation $y = 3x^2$

Solution: $\frac{dy}{dx}(3\,x^2) = 3 * 2x = 6x$

Derivative Sum Rule

If u and v are differentiable functions of x, then their sum $u + v$ is differentiable at every point where u and v are both differentiable. At such points,

$$\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}.$$

Example 6: Find the derivative of the polynomial $y = x^3 + (4/3)\,x^2 - 5x + 1$.

Solution: $\frac{dy}{dx} = \frac{d}{dx}x^3 + \frac{d}{dx}\left(\frac{4}{3}x^2\right) - \frac{d}{dx}(5x) + \frac{d}{dx}(1)$

$= 3x^2 + (4/3)*2x - 5 + 0$

$= 3x^2 + (8/3)*2x - 5$

Derivative Product Rule

If u and v are differentiable at x, then so is their product uv, and

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

Example 7: find the derivative of $y = (x^2 + 1)(x^3 + 3)$.

Solution:

(a) From the Product Rule we find:

$$\frac{d}{dx}\left[(x^2 + 1)(x^3 + 3)\right] = (x^2 + 1)(3x^2) + (x^3 + 3)(2x) \qquad \frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$= 3x^4 + 3x^2 + 2x^4 + 6x$$

$$= 5x^4 + 3x^2 + 6x.$$

(b) This particular product can be differentiated as well (perhaps better) by multiplying out the original expression for y and differentiating the resulting polynomial:

$$y = (x^2 + 1)(x^3 + 3) = x^5 + x^3 + 3x^2 + 3$$

$$\frac{dy}{dx} = 5x^4 + 3x^2 + 6x.$$

Derivative Quotient Rule

If u and v are differentiable at x and if $v(x) \neq 0$, then the quotient u/v is differentiable at x, and

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}.$$

Example 8: find the derivative of $y = \frac{t^2-1}{t^3+1}$ **Solution**: apply the Quotient Rule:

$$\frac{dy}{dt} = \frac{(t^3 + 1)\cdot 2t - (t^2 - 1)\cdot 3t^2}{(t^3 + 1)^2} \qquad \frac{d}{dt}\left(\frac{u}{v}\right) = \frac{v(du/dt) - u(dv/dt)}{v^2}$$

$$= \frac{2t^4 + 2t - 3t^4 + 3t^2}{(t^3 + 1)^2}$$

$$= \frac{-t^4 + 3t^2 + 2t}{(t^3 + 1)^2}.$$

Example 9: Find an equation for the tangent to the curve $y = x + 1/x$ at x = 2. **Solution**:
At $x = 2$:
$y = 2 + ½ = 5/2$ point
(2, 5/2)

$y = x + \frac{1}{x}$

$\frac{dy}{dx} = 1 + \frac{(x)(0)-(1)(1)}{x^2}$

$= 1 + \frac{-1}{x^2}$

At $x = 2$:

$\frac{dy}{dx} = m = 1 - \frac{1}{4} = ¾$

$(y - y_1) = m\ (x - x_1)$

$(y - 5/2) = 3/4\ (x - 2)$

$(2y - 5)/2 = 3(x - 2)/4$

$8y - 20 = 6(x - 2)$

$8y - 20 = 6x - 12$

$8y - 6x - 8 = 0$ y

$= 6/8\ x + 1$

Example 10: Find the point on the curve $y = x^3 + x^2 - 1$ where the tangent is parallel to the x- axis.

Solution:

Slope = $dy/dx = 3x^2 + 2x$

When the tangent is parallel to the x-axis, $m = 0$.

$3x^2 + 2x = 0$ x

$(3x + 2) = 0$

$x = 0$ or $3x + 2 = 0$ $x = -2/3$ at

$x = 0$ $y = -1$

$P_1\ (0, -1)$

At $x = -2/3 \quad y = -23/27$
$P_2 = (-2/3, -23/27)$

Example 11: Does the curve $y = x^4 - 2x^2 + 2$ have any horizontal tangents? If so, where?
Solution: The horizontal tangents, if any, occur where the slope dy/dx is zero. We have: $dy/dx = d/dx\,(x^4 - 2x^2 + 2)$

$= 4x^3 - 4x$ Now

solve the equation $dy/dx = 0$ for x:

$$4x^3 - 4x = 0$$
$$4x(x^2 - 1) = 0 \; x=$$
$$0, 1, -1$$

The curve *y = x4 – 2x2 + 2* has horizontal tangents at $x = 0$, 1 and -1. The corresponding points on the curve are (0, 2), (1, 1) and (-1, 1)

3.6 Derivatives of Trigonometric Functions

3.6.1 Derivative of the Sine Function

$$\sin(x + h) = \sin x \cos h + \cos x \sin h.$$

If $f(x) = \sin x$, then

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{\sin(x + h) - \sin x}{h} \qquad \text{Derivative definition}$$

$$= \lim_{h \to 0} \frac{(\sin x \cos h + \cos x \sin h) - \sin x}{h} = \lim_{h \to 0} \frac{\sin x (\cos h - 1) + \cos x \sin h}{h}$$

$$= \lim_{h \to 0} \left(\sin x \cdot \frac{\cos h - 1}{h}\right) + \lim_{h \to 0} \left(\cos x \cdot \frac{\sin h}{h}\right)$$

$$= \sin x \cdot \underbrace{\lim_{h \to 0} \frac{\cos h - 1}{h}}_{\text{limit } 0} + \cos x \cdot \underbrace{\lim_{h \to 0} \frac{\sin h}{h}}_{\text{limit } 1} = \sin x \cdot 0 + \cos x \cdot 1 = \cos x.$$

Example 5a and Theorem 7, Section 2.4

The derivative of the sine function is the cosine function:

$$\frac{d}{dx}(\sin x) = \cos x.$$

> **The derivative of the cosine function is the negative of the sine function:**
>
> $$\frac{d}{dx}(\cos x) = -\sin x.$$

Example 12: find the derivatives of the following functions:

(a) $y = \sin x \cos x$

(b) $y = \frac{cosx}{1-sinx}$

Solution:

(a)

$y = \sin x \cos x:$

$$\frac{dy}{dx} = \sin x \frac{d}{dx}(\cos x) + \cos x \frac{d}{dx}(\sin x)$$
$$= \sin x(-\sin x) + \cos x(\cos x)$$
$$= \cos^2 x - \sin^2 x$$

(b)

$y = \frac{\cos x}{1 - \sin x}:$

$$\frac{dy}{dx} = \frac{(1 - \sin x)\frac{d}{dx}(\cos x) - \cos x \frac{d}{dx}(1 - \sin x)}{(1 - \sin x)^2}$$
$$= \frac{(1 - \sin x)(-\sin x) - \cos x(0 - \cos x)}{(1 - \sin x)^2}$$
$$= \frac{1 - \sin x}{(1 - \sin x)^2}$$
$$= \frac{1}{1 - \sin x}$$

3.6.2 Derivatives of the Other Basic Trigonometric Functions

The derivatives of the other trigonometric functions:

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

Example 13: derive the following equations:

$y = \cos x \tan 3x$

$y = \tan\sqrt{3x}$ y

$= sin^2(\frac{1}{x})$

Solution:

$dy/dx = \cos x\ (\sec^2 3x *3) + \tan 3x\ (-\sin x)$

$= 3 \cos x \sec^2 3x - \sin x \tan 3x$

$dy/dx = \sec^2 (3x)^{1/2} * ½\ (3x)^{-1/2} *3$

$= \frac{3}{2\sqrt{3x}}\ sec^2\sqrt{3x}$

$y = (\sin\left(\frac{1}{x}\right))^2$

$dy/dx = 2\ (\sin\left(\frac{1}{x}\right) \times \cos\left(\frac{1}{x}\right) \times (-1 \times x^{-2}))\ dy/dx$

$= 2(\sin\left(\frac{1}{x}\right) \times \cos\left(\frac{1}{x}\right) \times (\frac{-1}{x^2})$

$= \frac{-2}{x^2} \sin\frac{1}{x} \cos\frac{1}{x}$

Example 14: find the point on the curve $y = \tan x$, $-\pi/2 < x < \pi/2$, where the tangent is parallel to the line $y = 2x$

Solution:

Slope of the line $y = 2x$ is $dy/dx = 2$

Slope of the curve $y = \tan x$ should be equal to 2 (parallel to line $y = 2x$)

$dy/dx = \sec^2 x = \frac{1}{cos^2 x}$

$$\frac{1}{cos^2 x} = 2$$

$\cos^2 x = ½$

$\cos x = \pm\sqrt{\frac{1}{2}} = \pm\frac{1}{\sqrt{2}}$

If $\cos x = -\frac{1}{\sqrt{2}}$ x out of interval (-π/2, π/2)

If $\cos x = \frac{1}{\sqrt{2}}$ x= π/4 and x = - π/4

For x = π /4 y = tan π/4 = 1

For x = - π /4 y = tan - π/4 = -1

The points are (π /4, 1) and (- π/4, -1)

3.7 The Chain Rule

The derivative of the composite function $f(g(x))$ at x is the derivative of f at $g(x)$ times the derivative of g at x. This is known as the Chain Rule (Figure 4).

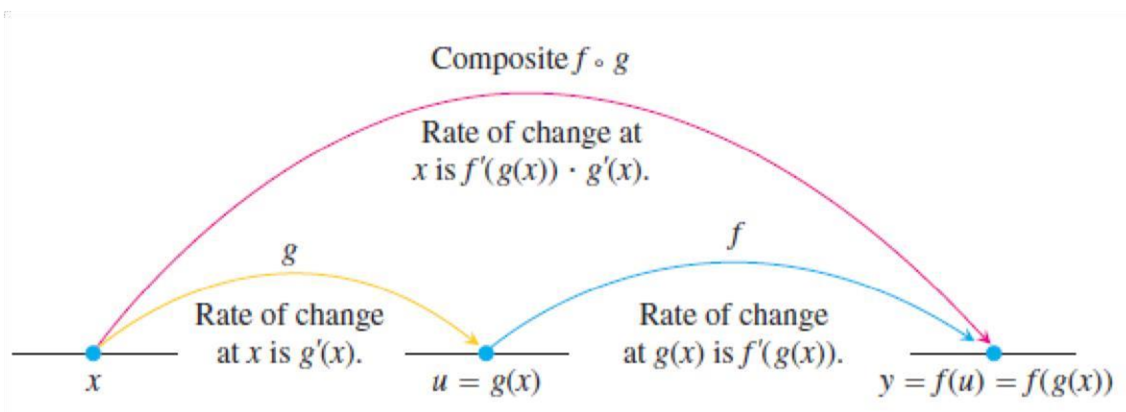


Figure 4

THEOREM 2—The Chain Rule If $f(u)$ is differentiable at the point $u = g(x)$ and $g(x)$ is differentiable at x, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x, and

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x).$$

In Leibniz's notation, if $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx},$$

where dy/du is evaluated at $u = g(x)$.

Example 15: If $y = x^2 + 2x + 1$, $x = 3u^2 + 1$, find dy/du.

Solution:
Method 1: substitute x function in y function:
$y = (3u^2 + 1)^2 + 2(3u^2 + 1) + 1 \quad = 9u^4 + 6u^2 +$
$1 + 6u^2 + 2 + 1$
$= 9u^4 + 12u^2 + 4$
$dy/dx = 36u^3 + 24u$

Method 2: Chain Rule
$dy/du = dy/dx * dx/du \ dy/dx$
$= 2x + 2, \quad dx/du = 6u$
$dy/du = (2x + 2)\,(6u) = [\,2(3u^2 + 1) + 2]\,(6u)$
$= (6u^2 + 4)\,(6u)$
$= (36u^3 – 24u)$

Example 16: Find dy/dt for $y = \sin(t^2 + 6)$ by using Chain Rule **Solution**:
Let $y = \sin u$ and $u = t^2 + 6$
$dy/dt = dy/du * du/dt \ dy/du$
$= \cos u, \quad du/dt = 2t \ dy/dx$
$= \cos u * 2t \qquad = \cos (t^2$
$+ 6) * 2t$
$= 2t \cos(t^2 + 6)$

3.8 Repeated Use of the Chain Rule

We sometimes have to use the Chain Rule two or more times to find a derivative.

Example 17: Find the derivative of function $g(t) = \tan(5 - \sin 2t)$.
Solution:
Notice here that the tangent is a function of $5 - \sin 2t$ whereas the sine is a function of $2t$, which is itself a function of t. Therefore, by the Chain Rule:

$$\begin{aligned} g'(t) &= \frac{d}{dt}(\tan(5 - \sin 2t)) \\ &= \sec^2(5 - \sin 2t) \cdot \frac{d}{dt}(5 - \sin 2t) \\ &= \sec^2(5 - \sin 2t) \cdot \left(0 - \cos 2t \cdot \frac{d}{dt}(2t)\right) \\ &= \sec^2(5 - \sin 2t) \cdot (-\cos 2t) \cdot 2 \\ &= -2(\cos 2t)\sec^2(5 - \sin 2t). \end{aligned}$$

Example 18: Show that the slope of every line tangent to the curve $y = 1/(1 - 2x)^3$ is positive.
Solution We find the derivative:

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx}(1 - 2x)^{-3} \\ &= -3(1 - 2x)^{-4} \cdot \frac{d}{dx}(1 - 2x) \\ &= -3(1 - 2x)^{-4} \cdot (-2) \\ &= \frac{6}{(1 - 2x)^4}. \end{aligned}$$

At any point (x, y) on the curve, $x \neq ½$ and the slope of the tangent line is :

$$\frac{dy}{dx} = \frac{6}{(1 - 2x)^4}$$

the quotient of two positive numbers.

3.9 Implicit Differentiation

Most of the functions we have dealt with so far have been described by an equation of the form $y = f(x)$ that expresses y explicitly in terms of the variable x. We have learned rules for differentiating functions defined in this way. Another situation occurs when we encounter equations like

$$x^3 + y^3 - 9xy = 0,\ y^2 - x = 0 \text{ or } x^2 + y^2 - 25 = 0.$$

These equations define an **implicit** relation between the variables x and y. In some cases we may be able to solve such an equation for y as an explicit function (or even several functions) of x. When we cannot put an equation $F(x, y) = 0$ in the form $y = f(x)$ to differentiate it in the usual way, we may still be able to find dy/dx by **implicit differentiation**. This section describes the technique.

Implicit Differentiation

1. Differentiate both sides of the equation with respect to x, treating y as a differentiable function of x.
2. Collect the terms with dy/dx on one side of the equation and solve for dy/dx.

Example 19:
(a) If $x^2 + y^2 = 25$, find dy/dx.
(b) Find an equation of the tangent to the circle $x^2 + y^2 = 25$ at the point (3, 4). **Solution**:

(a) Differentiate both sides of the equation $x^2 + y^2 = 25$

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(25)$$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = 0$$

Remembering that y is a function of x and using the Chain Rule, we have

$$\frac{d}{dx}(y^2) = \frac{d}{dy}(y^2)\frac{dy}{dx} = 2y\frac{dy}{dx}$$

Thus
$$2x + 2y\frac{dy}{dx} = 0$$

Now we solve this equation for dy/dx:

$$dy/dx = -x/y$$

(b) At the point (3, 4) we have $x = 3$ and $y = 4$, so

$$dy/dx = -3/4$$

An equation of the tangent to the circle at (3, 4) is therefore
$y - 4 = -3/4\,(x - 3)$ or $3x + 4y = 25$

Example 20: Find dy/dx if $y^2 = x^2 + \sin xy$.
Solution We differentiate the equation implicitly.

$$y^2 = x^2 + \sin xy$$

$$\frac{d}{dx}\left(y^2\right) = \frac{d}{dx}\left(x^2\right) + \frac{d}{dx}\left(\sin xy\right)$$ Differentiate both sides with respect to x ...

$$2y\frac{dy}{dx} = 2x + (\cos xy)\frac{d}{dx}\left(xy\right)$$ ... treating y as a function of x and using the Chain Rule.

$$2y\frac{dy}{dx} = 2x + (\cos xy)\left(y + x\frac{dy}{dx}\right)$$ Treat xy as a product.

$$2y\frac{dy}{dx} - (\cos xy)\left(x\frac{dy}{dx}\right) = 2x + (\cos xy)y$$ Collect terms with dy/dx.

$$(2y - x\cos xy)\frac{dy}{dx} = 2x + y\cos xy$$

$$\frac{dy}{dx} = \frac{2x + y\cos xy}{2y - x\cos xy}$$ Solve for dy/dx.

3.10 Derivatives of Higher Order

Example21: Find d^2y/dx^2 if $2x^3 - 3y^2 = 8.$

Solution:
To start, we differentiate both sides of the equation with respect to x in order to find $y = dy/dx$.

$$\frac{d}{dx}(2x^3 - 3y^2) = \frac{d}{dx}(8)$$

$$6x^2 - 6yy' = 0$$ Treat y as a function of x.

$$y' = \frac{x^2}{y}, \quad \text{when } y \neq 0$$ Solve for y'.

We now apply the Quotient Rule to find y''.

$$y'' = \frac{d}{dx}\left(\frac{x^2}{y}\right) = \frac{2xy - x^2y'}{y^2} = \frac{2x}{y} - \frac{x^2}{y^2}\cdot y'$$

Finally, we substitute $y' = x^2/y$ to express y'' in terms of x and y.

$$y'' = \frac{2x}{y} - \frac{x^2}{y^2}\left(\frac{x^2}{y}\right) = \frac{2x}{y} - \frac{x^4}{y^3}, \quad \text{when } y \neq 0$$ ■

Example 22: Show that the point (2, 4) lies on the curve $x^3 + y^3 - 9xy = 0$. Then find the tangent and normal to the curve there (Figure 5).

Solution: The point (2, 4) lies on the curve because its coordinates satisfy the equation given for the curve:

$$2^3 + 4^3 - 9\,(2)\,(4) = 8 + 64 - 72 = 0$$

To find the slope of the curve at (2, 4), we first use implicit differentiation to find a formula for *dy/dx*:

$$x^3 + y^3 - 9xy = 0$$

$$\frac{d}{dx}(x^3) + \frac{d}{dx}(y^3) - \frac{d}{dx}(9xy) = \frac{d}{dx}(0)$$

$$3x^2 + 3y^2\frac{dy}{dx} - 9\left(x\frac{dy}{dx} + y\frac{dx}{dx}\right) = 0$$

Differentiate both sides with respect to x.

$$(3y^2 - 9x)\frac{dy}{dx} + 3x^2 - 9y = 0$$

Treat xy as a product and y as a function of x.

$$3(y^2 - 3x)\frac{dy}{dx} = 9y - 3x^2$$

$$\frac{dy}{dx} = \frac{3y - x^2}{y^2 - 3x}.$$

Solve for dy/dx.

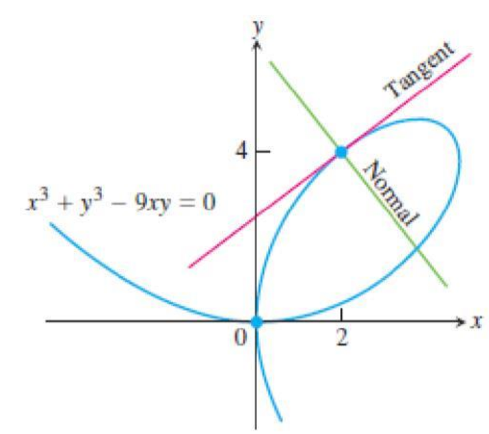


Figure 5

We then evaluate the derivative at $(x, y) = (2, 4)$:

$$\left.\frac{dy}{dx}\right|_{(2,4)} = \left.\frac{3y - x^2}{y^2 - 3x}\right|_{(2,4)} = \frac{3(4) - 2^2}{4^2 - 3(2)} = \frac{8}{10} = \frac{4}{5}.$$

The tangent at (2, 4) is the line through (2, 4) with slope 4/5:

$$y = 4 + \frac{4}{5}(x - 2)$$

$$y = \frac{4}{5}x + \frac{12}{5}.$$

The normal to the curve at (2, 4) is the line perpendicular to the tangent there, the line through (2, 4) with slope – 5/4:

$$y = 4 - \frac{5}{4}(x - 2)$$

$$y = -\frac{5}{4}x + \frac{13}{2}.$$

3.11 Parametric Equations

If $x = f(t)$ and $y = g(t)$, then these equations are called parametric equations and the variable t is called parameter.

From Chain Rule: $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$ $\quad \frac{dy}{dx} = \frac{\frac{dy}{du}}{\frac{dx}{du}}$

$x = f(t), y = g(t)$

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$ the 1 st derivative for parametric equation

For second derivative:

$\frac{d^2y}{dx^2} = \frac{dy/dt}{dx/dt}, \; y = \frac{dy}{dx}$

Or $\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}(\frac{dy}{dx})}{dx/dt}$ the 2nd derivative for parametric equation

Example 23: if $y = 2t^3 + 3$, $x = t/(t-1)$, find dy/dx

Solution:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$dy/dt = 6\,t^2$

$dx/dt = \frac{(t-1)(1)-t(1)}{(t-1)^2} = \frac{-1}{(t-1)^2}$

$dy/dx = \frac{6t^2}{\frac{-1}{(t-1)^2}}$

$dy/dx = -6t^2 (t-1)^2$

Example 24: If a point traces the circle $x^2 + y^2 = 25$ and if $dx/dt = 4$ when the point reaches (3, 4). Find dy/dt

Solution:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$x^2 + y^2 = 25$ $2x + 2y\,(dy/dx) = 0$ $dy/dx = -x/y$ At point (3, 4) $dy/dx = -3/4$

$-3/4 = \frac{dy/dt}{4}$ $dy/dt = -3$

Example 25: If $x = \cos 3t$, $y = \sin^2 3t$, find dy/dx, d^2y/dx^2 **Solution**:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$dy/dt = 2 \sin 3t (\cos 3t).3 = 6 \sin 3t \cos 3t$ $dx/dt = -\sin 3t .3 = -3 \sin 3t$

$\frac{dy}{dx} = \frac{6 \sin 3t \cos 3t}{-3 \sin 3t} = -2 \cos 3t = -2x$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt}$$

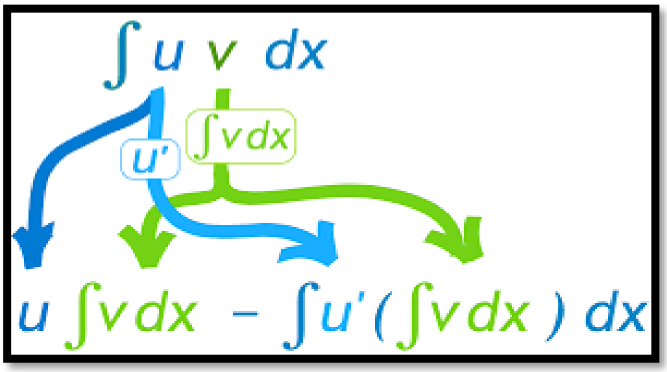
$\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{d}{dt}(-2\cos 3t) = -2(-\sin 3t).3$

$\frac{d^2y}{dx^2} = \frac{-2(-\sin 3t).3}{-3 \sin 3t} = -2$

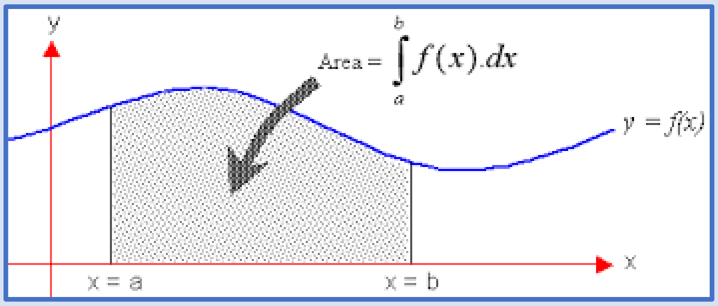
Or $dy/dx = -2x$

$d^2y/dx^2 = -2$ when dy/dx with respect to x.

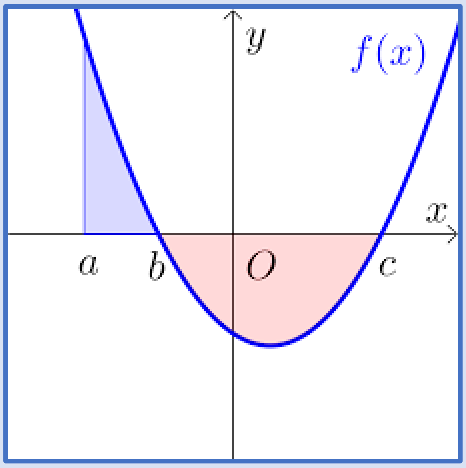
Lecture Thirteen



Behavioral objectives



1. Understand the concept of integration as the area under a curve.
2. Apply definite integrals to compute the area bounded by curves and the x-axis.
3. Use curve tracing techniques to determine symmetry and properties of functions.
4. Solve linear programming problems graphically.



AREA UNDER THE CURVE AND LINEAR PROGRAMMING

AREA UNDER THE CURVE

INTRODUCTION

In the previous chapters we have studied the process of integration and its physical interpretation. The most important application of integration is finding the area under a curve. In this topic we will discuss different curves and the area bounded by some simple plane curves taken together. In order to find the area, we need to know the basics of plotting a curve and then use integration with appropriate limits to get the answer. The process of finding area of some plane region is called **Quadrature**.

CURVE TRACING

Let us now discuss the basics of curve tracing. Curve tracing is a technique which provides a rough idea about the nature and shape of a plane curve. Different techniques are used in order to understand the nature of the curve, but there is no fixed rule which provides all the information to draw the graph of a given function (say f(x)). Sometimes it is also very difficult to draw the exact curve of the given function. However, the following steps can be helpful in trying to understand the nature and the shape of the curve.

Step 1: Check whether the origin lies on the given curve. Also check for other points lying on the curve by putting some values.

Step 2: Check whether the curve is increasing or decreasing by finding the derivative of the function. Also check for the boundary points of the curve.

Step 3: Check whether the curve $f(x, y) = 0$ is symmetric about

(a) X-axis: If the equation remains same on replacing y by –y i.e. $f(x, y) = f(x, -y)$, or, if all the powers of "y" are even, then the graph is symmetric about the X-axis.

(b) Y-axis: If the equation remains same on replacing x by –x i.e. $f(x, y) = f(-x, y)$, or, if all the powers of "x" are even, then the graph is symmetric about the Y-axis.

(c) Origin: If $f(-x, -y) = -f(x, y)$, then the graph is symmetric about the Origin.

For example, the curve given by $x^2 = y+2$ is symmetrical about y-axis, $y^2 = x+2$ is symmetrical about x-axis and the curve $y = x^3$ is symmetrical about the origin.

Step 4: Find out the points of intersection of the curve with the x-axis and y-axis by substituting y = 0 and x = 0 respectively.

For example, the curve $\frac{x^2}{9} + \frac{y^2}{4} = 1$ intersects the axes at points (± 3, 0) and (0, ±2).

Step 5: Identify the domain of the given function and the region in which the graph can be drawn. For example, the curve $xy^2 = (8 - 4x)$ or $y = 2\sqrt{\frac{2-x}{x}}$.

Therefore the value of y is defined only when $\frac{2-x}{x} \geq 0$ i.e. $0 < x \leq 2$. Hence, the graph lies between the lines x = 0 and x = 2.

Step 6: Check the behaviour of the graph as $x \to +\infty$ and as $x \to -\infty$. Find all the horizontal, vertical and oblique asymptotes, if any.

Step 7: Determine the critical points, the intervals on which the function (f) is concave up or concave down and the inflection points.

The information obtained from the Steps 1 to 7 are used to trace the curve.

Illustration 1: Trace the curve $y^2 (2a - x) = x^3$ **(JEE MAIN)**

Sol: By using curve tracing method as mentioned above.

Given curve: $y^2 = x^3/(2a - x)$ ...(i)

(a) Origin: The point (0 , 0) satisfies the given equation, therefore, it passes through the origin.

(b) Symmetrical about x-axis: On replacing y by –y, the equation remains same, therefore, the given curve is symmetrical about x-axis.

(c) Tangent at the origin: Equation of the tangent is obtained by equating the lowest degree terms to zero.

$\Rightarrow 2ay^2 = 0 \quad \Rightarrow y^2 = 0 \Rightarrow y=0$

(d) Asymptote parallel to y-axis: Equation of asymptote is obtained by equating the coefficient of lowest degree of y to 0. The given equation can be written as $(2a - x) y^2 = x^3$

$\therefore$ Equation of asymptote is $2a - x = 0$ or $x = 2a$

(e) Region of absence of curve: The given equation is

$y^2 (2a - x) = x^3 \quad \Rightarrow y^2 = \frac{x^3}{(2a-x)}$.

For x < 0 and x > 2a, RHS becomes negative, therefore the curve exists only for $0 \leq x < 2a$.

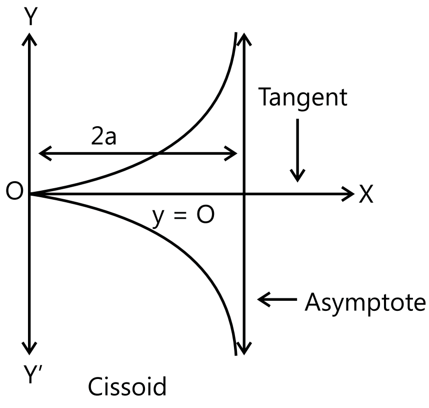


Figure 25.1

Hence the graph of $y^2 (2a - x) = x^3$ is as shown in Fig. 25.1. Such a curve is known as a Cissoid.

Illustration 2: Sketch the curve $\frac{x^2}{4} + \frac{y^2}{9} = 1$ **(JEE MAIN)**

Sol: Same as above illustration.

We have, $\frac{x^2}{4} + \frac{y^2}{9} = 1$ ...(i)

(a) Origin: The point (0,0) does not satisfy the equation, hence, the curve does not pass through O.

(b) Symmetry: The equation of the curve contains even powers of x and y so it is symmetric about both x and y axes.

(c) Intercepts: Putting y = 0, we get x = ± 2 i.e. the curve passes through the points (2 , 0) and (-2 , 0). Similarly, on substituting x = 0, we get y = ± 3 i.e. the curve passes through the points (0 , 3) and (0 , -3).

(d) Region where the curve does not exist: If $x^2 > 4$, y becomes imaginary. So the curve does not exist for x > 2 and x < –2. Similarly, if $y^2 > 9$, x becomes imaginary. So, the curve does not exist for y > 3 and y < –3.

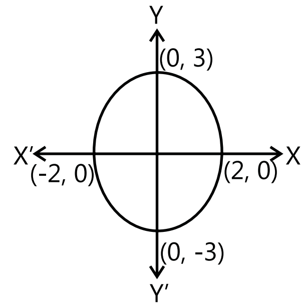


Figure 25.2

(e) Table:

x	– 2	0	1	2
y	0	±3	±2.6	0

Hence the graph of $\frac{x^2}{4}+\frac{y^2}{9}=1$ is as shown in Fig. 25.2.

MASTERJEE CONCEPTS

Using the above rules try to trace the Witch of Agnesi

$xy^2 = a^2 (a – x)$.

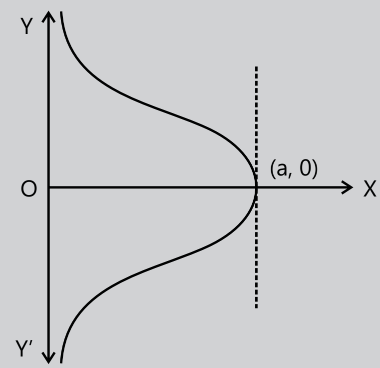


Figure 25.3

Vaibhav Krishnan (JEE 2009 AIR 22)

3. AREA BOUNDED BY A CURVE

3.1 The Area Bounded by a Curve with X-axis

The area bound the curve y=f(x) with the x-axis between the ordinates

x= a and x=b is given by $\text{Area} = \int_a^b y\,dx = \int_a^b f(x)\,dx$

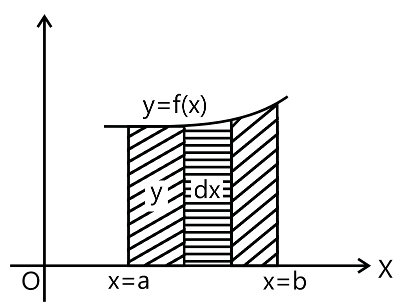


Figure 25.4: Area Bounded By a curve y=f(x) with x-axis

Illustration 3: Find the area bounded by the curve $y = x^3$, x-axis and ordinates x = 1 and x = 2. **(JEE MAIN)**

Sol: By using above formula, we can find out the area under given curve.

$$\text{Required Area} = \int_1^2 y\,dx = \int_1^2 x^3\,dx = \left(\frac{x^4}{4}\right)_1^2 = \frac{15}{4}$$

Illustration 4: Find the area bounded by the curve y = mx x-axis and ordinates x = 1 and x = 2. **(JEE MAIN)**

Sol: Same as above.

$$\text{Required area} = \int_1^2 y\,dx = \int_1^2 mx\,dx = \left(\frac{mx^2}{2}\right)_1^2 = \frac{m}{2}(4-1) = \frac{3}{2}m$$

Illustration 5: Find the area included between the parabola $y^2 = 4ax$ and its latus rectum (x = a). **(JEE ADVANCED)**

Sol: Here the curve is $y^2 = 4ax$, latus rectum is x = a, and the curve is symmetrical about the x-axis.

(a) The latus rectum is the line perpendicular to the axis of the parabola and passing through the focus S (a, 0).

(b) The parabola is symmetrical about the x-axis.

∴ The required area AOBSA = 2 × area AOSA

$$= 2\int_0^a y\,dx = 2\int_0^a 2\sqrt{ax}\,dx \quad \left[y^2 = 4ax \Rightarrow y = 2\sqrt{ax}\right]$$

$$= 4\sqrt{a}.\frac{2}{3}\left[x^{3/2}\right]_0^a = \frac{8}{3}\sqrt{a}.a^{3/2} = \frac{8}{3}a^2.$$

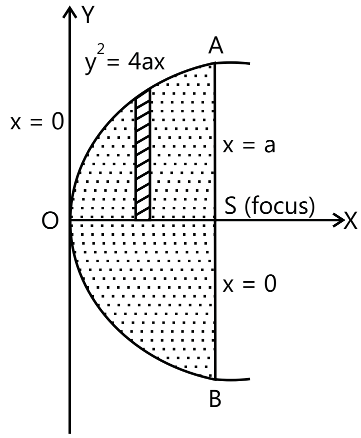


Figure 25.5

Illustration 6: Sketch the region {(x, y): $4x^2 + 9y^2 = 36$} and find its area using integration. **(JEE ADVANCED)**

Sol: The given curve is an ellipse, where a = 3 and b = 2. The X and Y axis divides this ellipse into four equal parts.

Region {(x, y): $4x^2 + 9y^2 = 36$} = Region bounded by $\left(\frac{x^2}{9} + \frac{y^2}{4} = 1\right)$

Limits for the shaded area are x = 0 and x = 3.

∴ The required area of the ellipse

$$= 4\int_0^a y\,dx = 4\int_0^3 2\sqrt{1-\frac{x^2}{9}}\,dx \quad \left[\because \frac{x^2}{9} + \frac{y^2}{4} = 1 \Rightarrow \frac{y^2}{4} = 1 - \frac{x^2}{9} \Rightarrow y = 2\sqrt{1-\frac{x^2}{9}}\right]$$

$$= \frac{8}{3}\int_0^3 \sqrt{3^2 - x^2}\,dx = \int_0^3 \frac{8}{3}\left[\frac{x}{2}\sqrt{9-x^2} + \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right)\right]_0^3 \quad \left[\text{using } \int\sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]$$

$$= \frac{8}{3}\left[0 + \frac{9}{2}\sin^{-1}1 - 0 - 0\right] = \frac{8}{3}\times\frac{9}{2}\times\frac{\pi}{2} = 6\pi \text{ sq. units.}$$

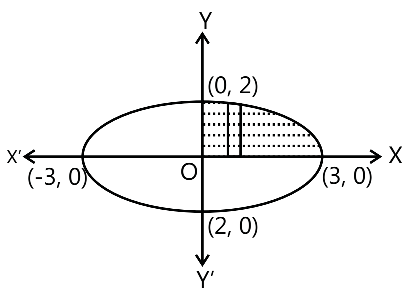


Figure 25.6

3.2 The Area Bounded by a Curve with y-Axis

The area bound the curve y=f(x) with y-axis between the ordinates

y= a and y=b is given by

$$\text{Area} = \int_c^d x\,dy = \int_c^d f(y)\,dy$$

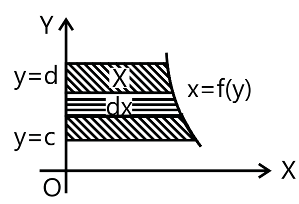


Figure 25.7: Area bounded by a curve with y-axis

Illustration 7: Find the area bounded by the curve $x^2 = \frac{1}{4}y$, y-axis and between the lines y = 1 and y = 4. **(JEE MAIN)**

Sol: As we know, area bounded by curve with y – axis is given by $\int_c^d x\,dy = \int_c^d f(y)\,dy$.

Required Area = $\int_1^4 x\,dy = 2\int_1^4 \frac{1}{2}\sqrt{y}\,dy = \frac{2}{3}\left[y^{3/2}\right]_1^4 = \frac{2}{3}(8-1) = \frac{14}{3}$ sq. units

Illustration 8: Find the area of the region bounded by the curve $y^2 = 4x$, y-axis and the line y = 3. **(JEE MAIN)**

Sol: Same as above illustration. $\left(\because y^2 = 4x \quad \frac{y^2}{4} = x\right)$

Area of region is $A = \int_{y=0}^{y=3} x\,dy = \int_0^3 \frac{y^2}{4}dy$

$= \frac{1}{4}\left[\frac{y^3}{3}\right]_0^3 = \frac{1}{4}\left[\frac{3^3}{3} - \frac{0}{3}\right] = \frac{1}{4}[9] = \frac{9}{4}$ sq. units

Hence, the required area is $\frac{9}{4}$ sq. units.

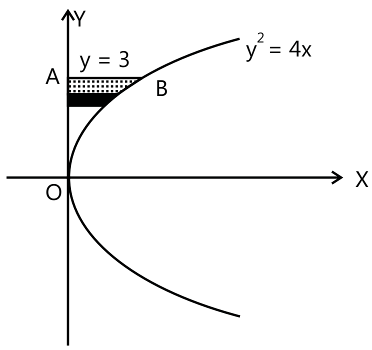


Figure 25.8

MASTERJEE CONCEPTS

There is no harm in splitting an integral into multiple components while finding area. If you have any doubt that the integral is changing sign, split the integral at that point.

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3.3 Area of a Curve in Parametric Form

If the given curve is in parametric form say x = f(t), y = g(t), then the area bounded by the curve with x-axis is equal to $\int_a^b y\,dx = \int_{t_2}^{t_1} g(t)f'(t)dt \quad \left[\because dx = d\left(f(t)\right) = f'(t)dt\right]$ Where t_1 and t_2 are the values of t corresponding to the values of a and b of x.

Illustration 9: Find the area bounded by the curve x = a cost, y = b sint in the first quadrant. **(JEE MAIN)**

Sol: Solve it using formula of area of a curve in parametric form.

The given equation is the parametric equation of ellipse, on simplifying we get $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

$\therefore$ Required area $= \int_0^a y\,dx = \int_{\pi/2}^0 (b\sin t(-a\sin t)dt) = ab\int_0^{\pi/2}\sin^2 t\,dt = \left(\frac{\pi ab}{4}\right)$.

3.4 Symmetrical Area

If the curve is symmetrical about a line or origin, then we find the area of one symmetrical portion and multiply it by the number of symmetrical portions to get the required area.

Illustration 10: Find the area bounded by the parabola $y^2 = 4x$ and its latus rectum. **(JEE MAIN)**

Sol: Here the given parabola is symmetrical about x – axis.

Hence required area $= 2\int_0^1 y\, dx$.

Since the curve is symmetrical about x-axis,

$\therefore$ The required Area $= 2\int_0^1 y\, dx \; = 2\int_0^1 \sqrt{4x}\, dx = 4 \cdot \frac{2}{3}\left[x^{3/2}\right]_0^1 = \frac{8}{3}$

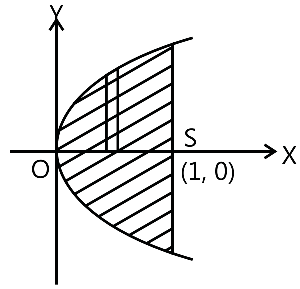

Figure 25.9

3.5 Positive and Negative Area

The area of a plane figure is always taken to be positive. If some part of the area lies above x-axis and some part lies below x-axis, then the area of two parts should be calculated separately and then add the numerical values to get the desired area.

If the curve crosses the x-axis at c (see Fig. 25.10), then the area bounded by the curve y = f(x) and the ordinates x = a and x = b, (b > a) is given by

$$A = \left|\int_a^c f(x)dx\right| + \left|\int_c^b f(x)dx\right|; \qquad A = \int_a^c f(x)dx - \int_c^b f(x)\, dx$$

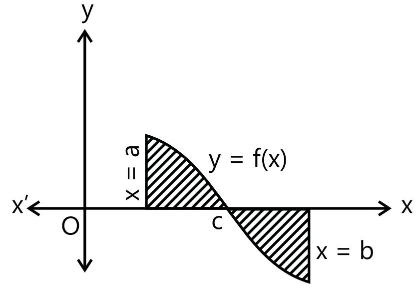

Figure 25.10

MASTERJEE CONCEPTS

To reduce confusion of using correct sign for the components, take modulus and add all the absolute values of the components.

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Illustration 11: Find the area between the curve y = cos x and x-axis when $\pi/4 < x < \pi$ **(JEE MAIN)**

Sol: Here some part of the required area lies above x-axis and some part lies below x-axis,. Hence by using above mentioned method we can obtain required area.

$\therefore$ Required area $= \int_{\pi/4}^{\pi/2} \cos x\, dx + \left| \int_{\pi/2}^{\pi} \cos x\, dx \right|$

$= [\sin x]_{\pi/4}^{\pi/2} + \left| [\sin x]_{\pi/2}^{\pi} \right| = \left(1 - 1/\sqrt{2}\right) + |0 - 1| = \frac{2\sqrt{2} - 1}{\sqrt{2}}$

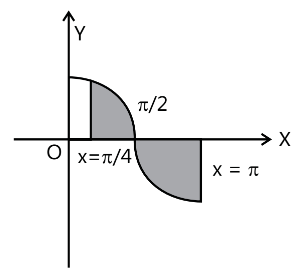


Figure 25.11

Illustration 12: Using integration, find the area of the triangle ABC, whose vertices are A (4, 1), B (6, 6) and C (8, 4) **(JEE ADVANCED)**

Sol: Here by using slope point form we can obtain respective equation of line by which given triangle is made. And after that by using integration method we can obtain required area.

Equation of line AB: $y - 1 = \frac{5}{2}(x - 4) \Rightarrow y = \frac{5x}{2} - 9$

Equation of line AC: $y - 1 = \left(\frac{3}{4}\right)(x - 4) \;\Rightarrow\; y = \frac{3x}{4} - 2$

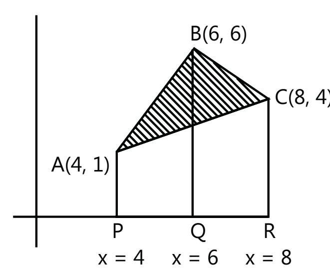


Figure 25.12

Equation of line BC: $(y-6)=\left(\frac{-2}{2}\right)(x-6) \Rightarrow y=-x+12$

∴ The required area = Area of trapezium ABQP + Area of trapezium BCRQ – Area of trapezium ACRP

$$=\int_4^6\left(\frac{5}{2}x-9\right)dx+\int_6^8(-x+12)dx-\int_4^8\left(\frac{3}{4}x-2\right)dx$$

$$=\left(\frac{5}{4}x^2-9x\right)_4^6+\left(12x-\frac{x^2}{2}\right)_6^8-\left(\frac{3}{8}x^2-2x\right)_4^8=7+10-10=7 \text{ sq. units.}$$

3.6 Area between Two Curves

(a) Area enclosed between two curves.

If $y=f_1(x)$ and $y=f_2(x)$ are two curves (where $f_1(x)>f_2(x)$), which intersect at two points, A (x = a) and B(x = b), then the area enclosed by the two curves between A and B is

$$\text{Common area} = \int_a^b (y_1-y_2)\,dx = \int_a^b [f_1(x)-f_2(x)]\,dx$$

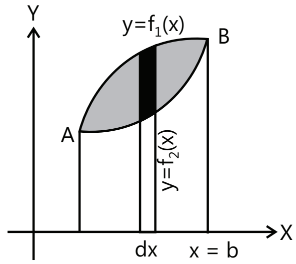


Figure 25.13

Illustration 13: Find the area between two curves $y^2=4ax$ and $x^2=4ay$. **(JEE MAIN)**

Sol: By using above mentioned formula of finding the area enclosed between two curves, we can obtain required area.

Given, $y^2=4ax$ ... (i)

$x^2=4ay$ ... (ii)

Solving (i) and (ii), we get x = 4a and y = 4a.

$$\text{So required area} = \int_0^{4a}\left(\sqrt{4ax}-\frac{x^2}{4a}\right)dx=\left(2\sqrt{a}\,\frac{x^{3/2}}{3/2}-\frac{x^3}{12a}\right)_0^{4a}$$

$$=\frac{4\sqrt{a}}{3}|4a|^{3/2}-\frac{64a^3}{12a}=\frac{16}{3}a^2$$

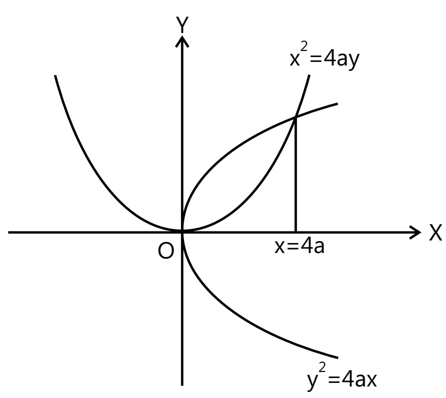


Figure 25.14

(b) Area enclosed by two curves intersecting at one point and the X-axis.

If $y=f_1(x)$ and $y=f_2(x)$ are two curves which intersect at a point P (α, β) and meet x-axis at A (a, 0) and B (b, 0) respectively, then the area enclosed between the curves and x-axis is given by

$$\text{Area} = \int_a^\alpha f_1(x)\,dx+\int_\alpha^b f_2(x)\,dx$$

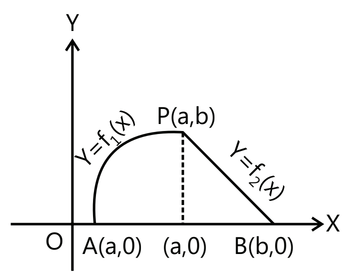


Figure 25.15

(c) Area bounded by two intersecting curves and lines parallel to y-axis.

The area bounded by two curves y = f(x) and y = g(x) (where a ≤ x ≤ b), when they intersect at x = c ∈ (a, b), is given by $A = \int_a^b |f(x) - g(x)|\,dx \Rightarrow A = \int_a^c (f(x) - g(x))\,dx + \int_c^b (g(x) - f(x))\,dx$

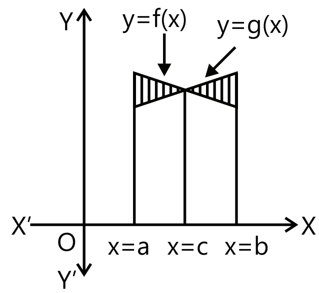


Figure 25.16

Illustration 14: Draw a rough sketch of the region enclosed between the circles $x^2 + y^2 = 4$ and $(x-2)^2 + y^2 = 4$. Using method of integration, find the area of this enclosed region **(JEE ADVANCED)**

Sol: By solving given equations simultaneously, we will be get intersection points of circles and then by using integration method we can obtain required area.

The figure shown alongside is the sketch of the circles

$x^2 + y^2 = 4$ ... (i)

and, $(x-2)^2 + y^2 = 4$ ... (ii)

From (i) and (ii), we have $(x-2)^2 - x^2 = 0$

$\Rightarrow (x - 2 - x)(x - 2 + x) = 0 \quad \Rightarrow x = 1$ ... (iii)

Solving (i) and (iii), we get $y = \pm\sqrt{3}$

Therefore, the circles (i) and (ii) intersect at $A(1, \sqrt{3})$ and $B(1, -\sqrt{3})$.

Area of enclosed region = Area OACBO = 2 Area OACO

= 2 [Area OAD + Area ACD]

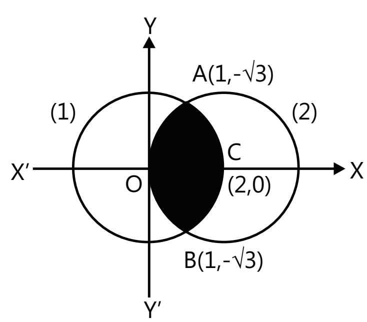


Figure 25.17

$$= 2\int_0^1 \sqrt{4-(x-2)^2}\,dx + 2\int_1^2 \sqrt{4-x^2}\,dx$$

$$= 2\int_1^2 \sqrt{4-x^2}\,dx + 2\int_0^1 \sqrt{4-(x-2)^2}\,dx$$

$$= 2\left[\frac{x\sqrt{4-x^2}}{2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_1^2 + 2\left[\frac{(x-2)\sqrt{4-(x-2)^2}}{2} + \frac{4}{2}\sin^{-1}\left(\frac{x-2}{2}\right)\right]_0^1 \quad \left[\because \sqrt{a^2-x^2}\,dx \Rightarrow \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]$$

$$= 2\left(\pi - \frac{\sqrt{3}}{2} - 2\left(\frac{\pi}{6}\right)\right) + 2\left(-\frac{\sqrt{3}}{2} - 2\left(\frac{\pi}{6}\right) + \pi\right) = \frac{8\pi}{3} - 2\sqrt{3} \text{ sq. units}$$

Illustration 15: Using integration, find the area of the region given below:

$\{(x, y): 0 \le y \le x^2 + 1,\ 0 \le y \le x + 1,\ 0 \le x \le 2\}$ **(JEE ADVANCED)**

Sol: Same as above illustration, by solving given equation $y = x^2 + 1$ and $y = x + 1$ we will be get their points of intersection and after that using integration method and taking these points as limit we can obtain required area.

The region is shaded as shown in the Fig. 25.18.

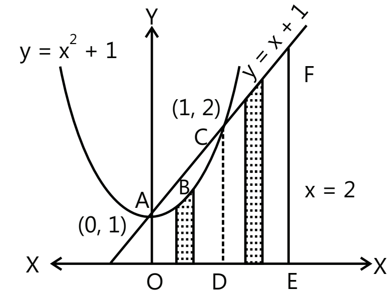


Figure 25.18

Given, $y = x^2 + 1$ ... (i)

$y = x + 1$ ... (ii)

On solving (i) and (ii), we have $x^2 + 1 = x + 1$

$\Rightarrow x = 0, 1$ and $y = 1, 2$

$\therefore$ The shaded region can be divided into two parts OABCDO and CDEFC.

Limits for the area OABEO are x = 0 and x = 1.

Limits for the area EBDFE are x = 1 and x = 2.

Area of the shaded region = Area OABEO + Area EBDFE.

$$= \int_0^1 (x^2+1)dx + \int_1^2 (x+1)dx = \left[\frac{x^3}{3}+x\right]_0^1 + \left[\frac{x^2}{2}+x\right]_1^2 = \left(\frac{1}{3}+1\right)+\left(\frac{4}{2}+2-\frac{1}{2}-1\right) = \frac{23}{6} \text{ sq. units}$$

Illustration 16: Find the area of the following region: $[(x, y): y^2 \le 4x, 4x^2+4y^2 \le 9]$ **(JEE ADVANCED)**

Sol: Similar to above problem, Here the required area is equal to Area AOBA + Area ACBA.

Given $y^2 = 4x$ ... (i)

$4x^2 + 4y^2 = 9 \Rightarrow x^2 + y^2 = \left(\frac{3}{2}\right)^2$ ... (ii)

Curves (i) and (ii) intersect at $A\left(\frac{1}{2}, \sqrt{2}\right)$ and $B\left(\frac{1}{2}, -\sqrt{2}\right)$

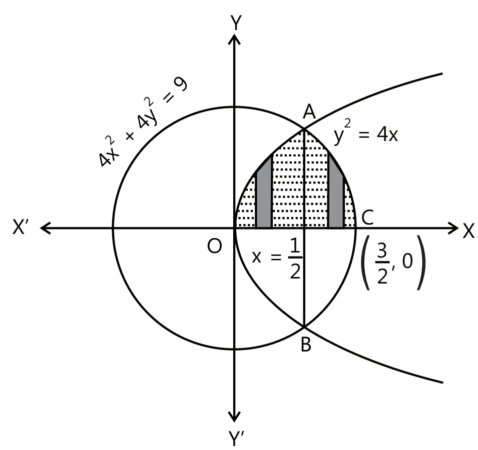


Figure 25.19

Limits for the area OAB are $x = 0,\ x = \frac{1}{2}$

Limits for the area ACB are $x = \frac{1}{2},\ x = \frac{3}{2}$.

The required area = Area AOBA + Area ACBA

$$= 2\left[\int_0^{1/2} y_1\, dx + \int_{1/2}^{3/2} y_2\, dx\right] = 2\left[\int_0^{1/2} \sqrt{4x}\, dx + \int_{1/2}^{3/2} \sqrt{\frac{9}{4}-x^2}\, dx\right]$$

$$= 4\left[\frac{2}{3}x^{3/2}\right]_0^{1/2} + 2\left[\frac{x}{2}\cdot\sqrt{\frac{9}{4}-x^2} + \frac{9}{8}\sin^{-1}\left(\frac{x}{3/2}\right)\right]_{1/2}^{3/2}$$

$$= \frac{8}{3}\cdot\frac{1}{2\sqrt{2}} + \left[0 - \frac{1}{\sqrt{2}} + \frac{9}{4}\left(\sin^{-1}1 - \sin^{-1}\frac{1}{3}\right)\right] = \frac{4}{3\sqrt{2}} - \frac{1}{\sqrt{2}} + \frac{9}{4}\left(\frac{\pi}{2} - \sin^{-1}\frac{1}{3}\right) = \frac{1}{3\sqrt{2}} + \frac{9}{4}\cos^{-1}\frac{1}{3}.$$

Illustration 17: Draw a rough sketch and find the area of the region bounded by the two parabolas $y^2 = 8x$ and $x^2 = 8y$, by using method of integration. **(JEE MAIN)**

Sol: As the given two equation is the equation of parabola which intersect at O(0, 0) and A(8, 8), and the required area is equal to Area OBADO – Area OADO.

Given parabolas are $y^2 = 8x$ ... (i)

and, $x^2 = 8y$ ... (ii)

The curves (i) and (ii) intersect at O(0, 0) and A(8, 8).

$\therefore$ Required Area = Area OBADO – Area OADO

$$= \int_0^8 (y_1 - y_2)\,dx$$

$$= \int_0^8 \left(\sqrt{8x} - \frac{x^2}{8}\right)dx = \left[2\sqrt{2}.\frac{x^{3/2}}{3/2} - \frac{1}{8}\frac{x^3}{3}\right]_0^8 = \frac{64}{3} \text{ sq. units.}$$

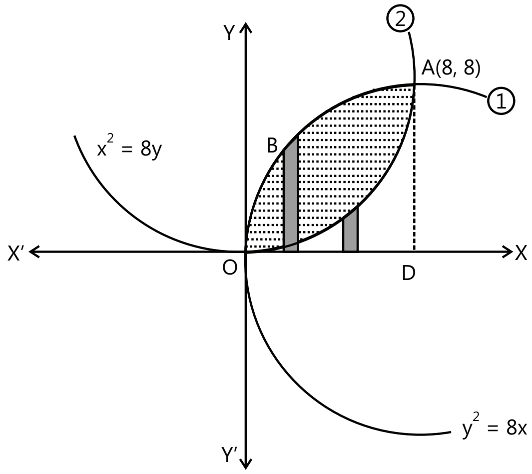


Figure 25.20

Illustration 18: Find the area between the curves y = 2x, x + y = 1 and x-axis. **(JEE MAIN)**

Sol: Here y = 2x and x + y = 1 is a two line intersect at $p\left(\frac{1}{3},\frac{2}{3}\right)$, therefore using integration method we can obtain required area.

Given y = 2x ... (i)

and, x + y = 1 ... (ii)

Solving (i) and (ii), we get x + 2x = 1 $\Rightarrow$ x = 1/3.

Line (i) intersects with the x – axis at the origin and the line (ii) intersects with the x – axis at x = 1.

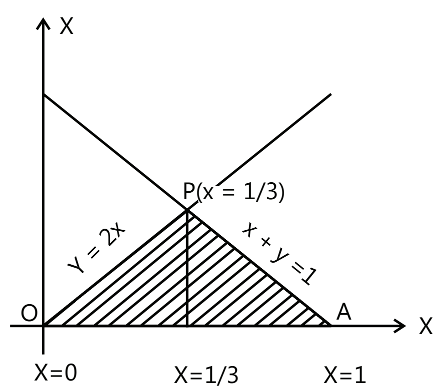


Figure 25.21

So required area = $\int_0^{1/3} 2x\,dx + \int_{1/3}^{1} (1-x)\,dx = \left[x^2\right]_0^{1/3} + \left(x - \frac{x^2}{2}\right)_{1/3}^{1}$

$$= \frac{1}{9} + \left(\frac{1}{2}\right) - \left(\frac{1}{3} - \frac{1}{18}\right) = \frac{1}{3} \text{ sq. units}$$

Illustration 19: Using the method of integration, find the area of the region bounded by lines: 2x + y = 4, 3x – 2 y = 6 and x – 3y + 5 = 0 **(JEE ADVANCED)**

Sol: Same as above problem.

Given equation of the lines are 2x + y = 4 ... (i)

$3x - 2y = 6$ ... (ii)

$x - 3y + 5 = 0$ ... (iii)

Solving (i) and (ii), we get (2, 0)

Solving (ii) and (iii), we get (4, 3)

Solving (i) and (iii), we get (1, 2)

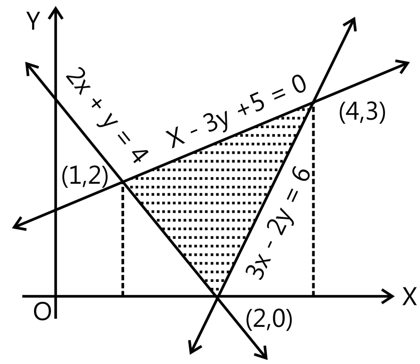


Figure 25.22

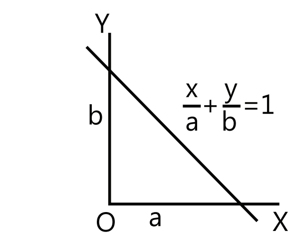
$\therefore$ Required Area = $\int_1^4 \left(\frac{x+5}{3}\right)dx - \int_1^2 (4-2x)\,dx - \int_2^4 \left(\frac{3x-6}{2}\right)dx$

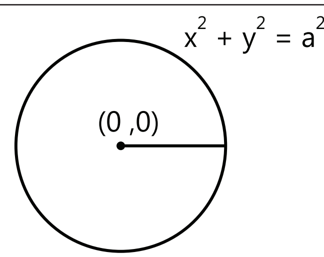
$$= \frac{1}{3}\left[\frac{x^2}{2}+5x\right]_1^4 - [4x - x^2]_1^2 - \frac{1}{2}\left[\frac{3x^2}{2}-6x\right]_2^4$$

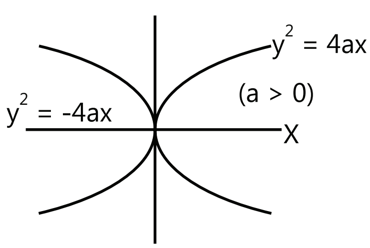
$$= \frac{1}{3}\left[(8+20)-\left(\frac{1}{2}+5\right)\right] - [(8-4)-(4-1)] - \frac{1}{2}[(24-24)-(6-12)]$$

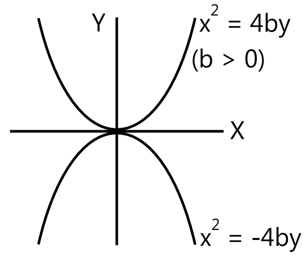
$$= = \frac{7}{2} \text{ sq. units.}$$

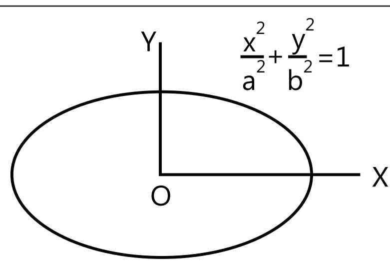
SKETCH OF STANDARD CURVES

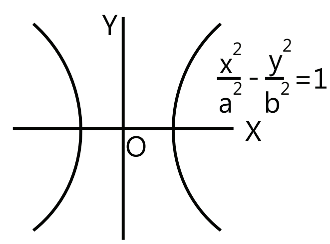


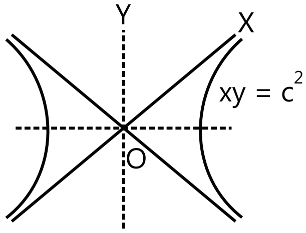


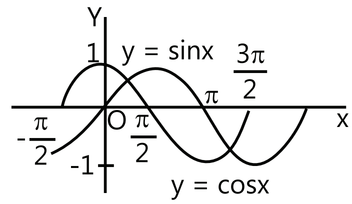


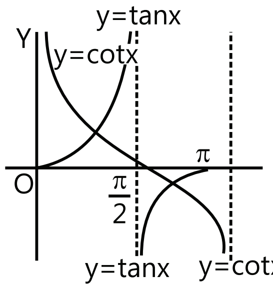


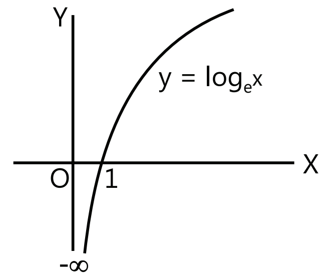


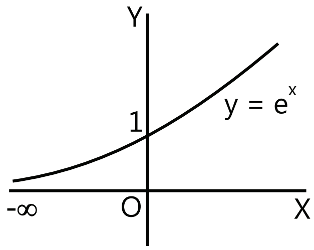












4. STANDARD AREAS

4.1 Area Bounded by Two Parabolas

Area between the parabolas $y^2 = 4ax$ and $x^2 = 4$; a > 0, b > 0, is

$$|A| = \frac{16ab}{3}$$

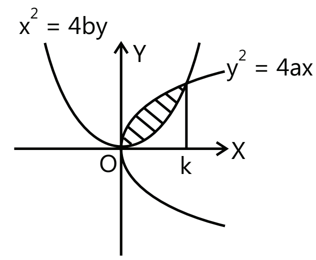


Figure 25.23

Illustration 20: Find the area bounded by $y = \sqrt{x}$ and $x = \sqrt{y}$. **(JEE MAIN)**

Sol: By using above mentioned formula.

Area bounded is shaded in the figure

Here, $a = \frac{1}{4}$ and $b = \frac{1}{4}$

$\therefore$ Using the above formula, Area = (16 ab)/ 3

$$= \frac{16 \times (1/4) \times (1/4)}{3} = \frac{1}{3}$$

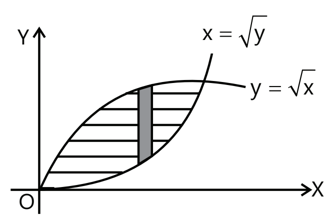


Figure 25.24

4.2 Area Bounded By Parabola and a Line

Area bounded by $y^2 = 4ax$ and $y = mx$; $a > 0$, $m > 0$ is $A = \frac{8a^2}{3m^3}$

Area bounded by $x^2 = 4ay$ and $y = mx$; $a > m > 0$

is $y = mx$; $a > m > 0$ $A = \frac{8a^2}{3m^3}$

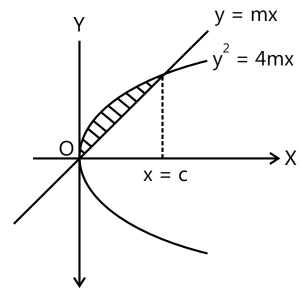


Figure 25.25

Illustration 21: Find the area bounded by, $x^2 = y$ and $y = |x|$. **(JEE MAIN)**

Sol: Using above formula, i.e. $A = \frac{8a^2}{3m^3}$

Area bounded is shaded in the Fig. 25.26.

Here, a = 1/4, m = 1

$\therefore$ Using the above formula, Area $= 2\left(\frac{8a^2}{3m^3}\right) = \frac{2\times 8\times\left(\frac{1}{4}\right)^2}{3\times(1)^3} = \frac{1}{3}$

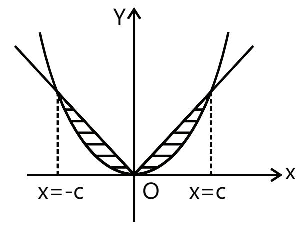


Figure 25.26

Illustration 22: Find the area bounded by $y^2 = x$ and $x = |\, y\,|$. **(JEE MAIN)**

Sol: Here, a = 1/4, m = 1, and required area is divided in to two equal parts at above and below x – axis.

Hence required area will be $2\left(\frac{8a^2}{3m^3}\right)$.

$\therefore$ Using the above formula, Area $= 2\left(\frac{8a^2}{3m^3}\right) = \frac{2\times 8\times(1/4)^2}{3\times(1)^3} = \frac{1}{3}$

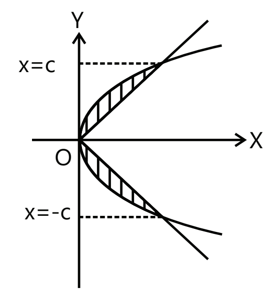


Figure 25.27

4.3 Area Enclosed by Parabola and It's Chord

Area between $y^2 = 4ax$ and its double ordinate at x = a is

Area of AOB = $\frac{2}{3}$ (area ▱ABCD)

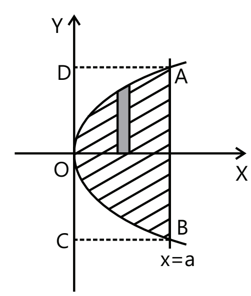


Figure 25.28

Illustration 23: Find the area bounded by $y = 2x - x^2$, $y + 3 = 0$. **(JEE MAIN)**

Sol: Here first obtain area of rectangle ABCD and after that by using above mentioned formula we will be get required area.

Solving $y = 2x - x^2$, $y + 3 = 0$, we get x = –1 or 3

Area (ABCD) = 4 × 4 = 16.

$\therefore$ Required area $= \frac{2}{3}\times 16 = \frac{32}{3}$

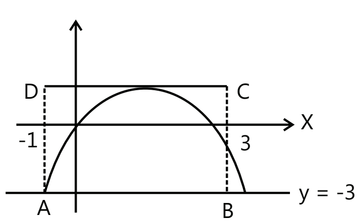


Figure 25.29

4.4 Area of an Ellipse

For an ellipse of the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is A = πab

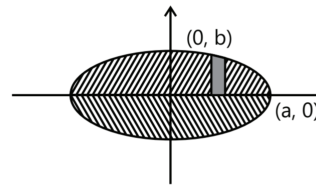


Figure 25.30

MASTERJEE CONCEPTS

Try to remember some standard areas like for ellipse, parabola. These results are sometimes very helpful.

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5. SHIFTING OF ORIGIN

Area remains unchanged even if the coordinate axes are shifted or rotated or both. Hence shifting of origin / rotation of axes in many cases proves to be very convenient in finding the area.
For example: If we have a circle whose centre is not origin, we can find its area easily by shifting circle's centre.

Illustration 24: The line 3x + 2y = 13 divides the area enclosed by the curve $9x^2 + 4y^2 - 18x - 16y - 11 = 0$ into two parts. Find the ratio of the larger area to the smaller area. **(JEE ADVANCED)**

Sol: Given $9x^2 + 4y^2 - 18x - 16y - 11 = 0$ ... (i)

and, 3x + 2y = 13 ... (ii)

$9(x^2 - 2x) + 4(y^2 - 4y) = 11;$

$\Rightarrow 9[(x-1)^2 - 1] + 4[(y-2)^2 - 4] = 11$

$\Rightarrow 9(x-1)^2 + 4(y-2)^2 = 36$

$\Rightarrow \frac{(x-1)^2}{4}+\frac{(y-2)^2}{9}=1 \Rightarrow \frac{X^2}{4}+\frac{Y^2}{9}=1$ (where X = x – 1 and Y = y – 2)

Hence 3x + 2y = 13

$\Rightarrow 3(X + 1) + 2(Y + 2) = 13$

$\Rightarrow 3X + 2Y = 6$

$\Rightarrow \frac{X}{2}+\frac{Y}{3}=1$

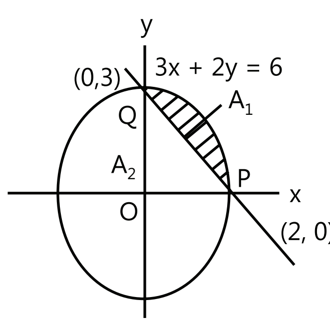


Figure 25.31

$\therefore$ Area of triangle OPQ = 1/2 × 2 × 3 = 3

Also area of ellipse = π (semi major axes) (semi minor axis) = π.3.2 = 6π

$$A_1 = \frac{6\pi}{4} - \text{area of } \Delta OPQ = \frac{3\pi}{2} - 3$$

$$A_2 = 3\left(\frac{6\pi}{4}\right) + \text{area of } \Delta OPQ = \frac{9\pi}{2} + 3$$

Hence, $$\frac{A_2}{A_1} = \frac{\frac{9\pi}{2}+3}{\frac{3\pi}{2}-3} = \frac{3\pi+2}{\pi-2}$$

6. DETERMINATION OF PARAMETERS

In this type of questions, you will be given area of the curve bounded between some axes or points, and some parameter(s) will be unknown either in equation of curve or a point or an axis. You have to find the value of the parameter by using the methods of evaluating area.

Illustration 25: Find the value of c for which the area of the figure bounded by the curves $y = \frac{4}{x^2}$; x = 1 and y = c is equal to $\frac{9}{4}$. **(JEE MAIN)**

Sol: By using method of evaluating area we can find out the value of c.

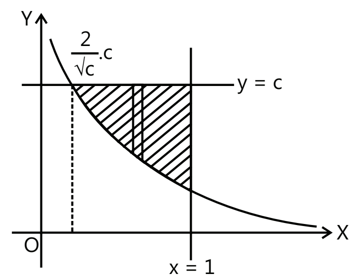


Figure 25.32

$$A = \int_{2/\sqrt{c}}^{1}\left(c - \frac{4}{x^2}\right)dx = \frac{9}{4}; \quad \left(cx + \frac{4}{x}\right)_{\frac{2}{\sqrt{c}}}^{1} = \frac{9}{4}$$

$$(c+4) - (2\sqrt{c} + 2\sqrt{c}) = \frac{9}{4}; \; c - 4\sqrt{c} + 4 = \frac{9}{4}$$

$$\Rightarrow \; (\sqrt{c} - 2)^2 = \frac{9}{4} \Rightarrow (\sqrt{c} - 2) = \frac{3}{2} \text{ or } -\frac{3}{2}$$

Hence c = (49/4) or (1/4)

Illustration 26: Consider the two curves:

C_1 : y = 1 + cos x, and C_2 : y = 1 + cos(x – α) for α ∈ (0, π/2) and x ∈ [0, p].

Find the value of α, for which the area of the figure bounded by the curves C_1, C_2 and x = 0 is same as that of the area bounded by C_2, y = 1 and x = π. For this value of α, find the ratio in which the line y = 1 divides the area of the figure by the curves C_1, C_2 and x = π. **(JEE ADVANCED)**

Sol: Solve C_1 and C_2 to obtain the value of x, after that by following given condition we will be obtain required value of α.

Solving C_1 and C_2 , we get

$$1 + \cos x = 1 + \cos(x - \alpha) \Rightarrow \; x = \alpha - x \qquad \Rightarrow x = \frac{\alpha}{2}$$

According to the question,

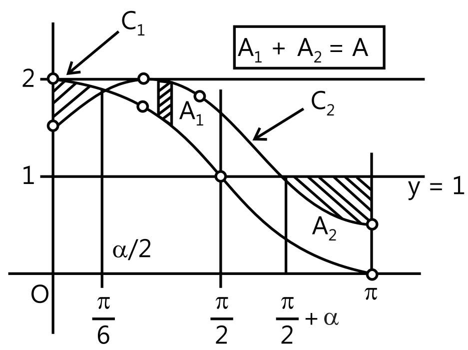


Figure 25.33

$$\int_0^{\alpha/2} (\cos x - \cos(x - \alpha))\,dx = -\int_{\frac{\pi}{2}+\alpha}^{\pi} (\cos(x - \alpha))\,dx$$

$$\Rightarrow \; \left[\sin x - \sin(x - \alpha)\right]_0^{\alpha/2} = \left[\sin(x - \alpha)\right]_{\pi}^{\frac{\pi}{2}+\alpha}$$

$$\Rightarrow \; \left[\sin\frac{\alpha}{2} - \sin\left(-\frac{\alpha}{2}\right)\right] - [0 - \sin(-\alpha)] \; = \; \sin\left(\frac{\pi}{2}\right) - \sin(\pi - \alpha)$$

$$\Rightarrow \; 2\sin\frac{\alpha}{2} - \sin\alpha = 1 - \sin\alpha \text{ . Hence, } 2\sin\frac{\alpha}{2} = 1 \qquad \Rightarrow \; \alpha = \frac{\pi}{3}$$

7. AREA BOUNDED BY THE INVERSE FUNCTION

The area of the region bounded by the inverse of a given function can also be calculated using this method. The graph of inverse of a function is symmetric about the line y = x. We use this property to calculate the area. Hence, area of the function between x = a to x = b, is equal to the area of inverse function from f(a) to f(b).

Illustration 27: Find the area bounded by the curve g(x), the x-axis and the lines at y = –1 and y = 4, where g(x) is the inverse of the function $f(x)=\frac{x^3}{24}+\frac{x^2}{8}+\frac{13x}{12}+1$. **(JEE MAIN)**

Sol: Here f(x) is a strictly increasing function therefore required area will be

$$A=\int_0^2 (4-f(x))dx+\int_{-2}^0 (f(x)+1)dx$$

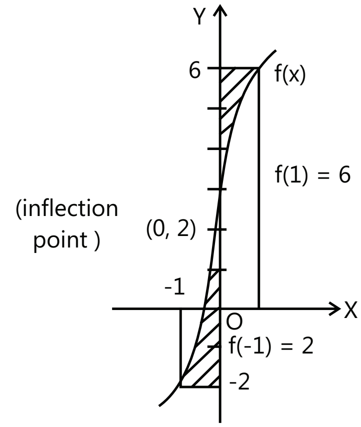


Figure 25.34

Given $f(x)=\frac{x^3}{24}+\frac{x^2}{8}+\frac{13x}{12}+1$

$\Rightarrow$ f(0) = 1; f(2) = 4 and f(–2) = –1

Also, $f'(x)=\frac{x^2}{8}+\frac{x}{4}+\frac{13}{12}$,

i.e. f(x) is a strictly increasing function.

$$\therefore A=\int_0^2 (4-f(x))dx+\int_{-2}^0 (f(x)+1)dx$$

$$A=\int_0^2\left(4-\frac{x^3}{24}-\frac{x^2}{8}-\frac{13x}{12}-1\right)dx+\int_{-2}^0\left(\frac{x^3}{24}+\frac{x^2}{8}+\frac{13x}{12}+1+1\right)dx$$

$$\therefore A=\left[\left(3.2-\frac{2^4}{24.4}-\frac{2^3}{8.3}-\frac{13.2^2}{12.2}\right)-(0)\right]+\left[(0)-\left(\frac{2^4}{24.4}-\frac{2^3}{8.3}+\frac{13.2^2}{12.2}-2.2\right)\right]=\frac{16}{3}$$

Illustration 28: Let f(x) = x^3 + 3x + 2 and g(x) is the inverse of it. Find the area bounded by g(x), the x-axis and the ordinate at x = –2 and x = 6. **(JEE ADVANCED)**

Sol: Let $A=\int_{-2}^{6}\left|f^{-1}(x)\right|dx$

Substitute $x=f(u)$ or $u=f^{-1}(x)$

$$=\int_{f^{-1}(2)}^{f^{-1}(6)}|u|\,f^{-1}(u)\,du$$

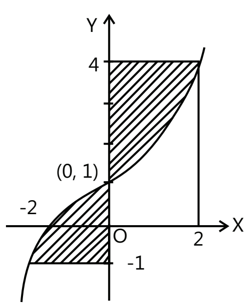


Figure 25.35

$$=\int_{f^{-1}(2)}^{f^{-1}(6)}|4|\left(3u^2+3\right)du$$

We have, $f(-1)=2$ and $f(1)=6$

$$=\int_{-1}^{1}|u|\left(3\,u^2+3\right)du=2\int_0^1\left(3\,u^3+3u\right)du$$

$$=\left[\frac{3}{2}u^4+3u^2\right]_0^1=\frac{9}{2}\text{ Sq. units.}$$

8. VARIABLE AREA

If y = f(x) is a monotonic function in (a, b), then the area of the function y = f(x) bounded by the lines at x = a, x = b, and the line y = f(c), [where c ∈ (a, b)] is minimum when $c = \frac{a+b}{2}$.

Proof: A = $\int_a^c f(c) - f(x)dx + \int_c^b (f(x) - f(c))dx$

$= f(c)(c-a) - \int_a^c (f(x))dx + \int_c^b (f(x))dx - f(c)(b-c)$

$= \{(c-a) - (b-c)\} f(c) + \int_c^b (f(x))dx - \int_a^c (f(x))dx$

$A = [2c - (a+b)]\, f(c) + \int_c^b (f(x))dx - \int_a^c (f(x))dx$

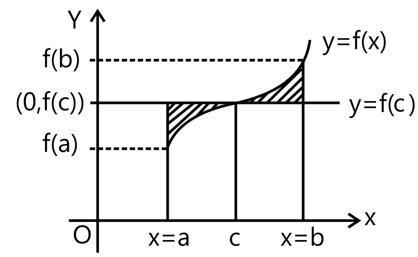


Figure 25.36

For maxima and minima $\frac{dA}{dc} = 0 \Rightarrow$ f ′ (c) = [2c – (a + b)] = 0 (as f ′ (c) = 0) hence $c = \frac{a+b}{2}$ also $c < \frac{a+b}{2}, \frac{dA}{dc} < 0$ and $c > \frac{a+b}{2}, \frac{dA}{dc} > 0$ Hence A is minimum when c = $\frac{a+b}{2}$

9. AVERAGE VALUE OF A FUNCTION

In this section, we would study the average of a continuous function. This concept of average is frequently applied in physics and chemistry.

Average of a function f(x) between x = a to x = b is given by $y_{av} = \frac{1}{b-a}\int_a^b f(x)dx$

MASTERJEE CONCEPTS

(a) Average value can be positive, negative or zero .

(b) If the function is defined in (0, ∞), then $y_{av} = \lim_{b \to \infty} \frac{1}{b}\int_0^b f(x)dx$ provided the limit exists

(c) Root mean square value (RMS) is defined as $\rho = \left[\frac{1}{b-a}\int_a^b f^2(x)dx\right]^{\frac{1}{2}}$

(d) If a function is periodic then we need to calculate average of function in particular time period that is its overall mean.

Vaibhav Krishnan (JEE 2009 AIR 22)

Illustration 29: Find the average value of y^2 w.r.t. x for the curve ay = b$\sqrt{a^2 - x^2}$ between x = 0 & x = a. Also find the average value of y w.r.t. x^2 for $0 \le x \le a$. **(JEE MAIN)**

Sol: As average of a function f(x) between x = a to x = b is given by $y_{av} = \frac{1}{b-a}\int_a^b f(x)dx$

Let $f(x) = y^2 = \frac{b^2}{a^2}(a^2 - x^2)$ Now $f(x)|_{av} = \frac{b^2}{a^2(a-0)}\int_0^a (a^2 - x^2)dx = \frac{2b^2}{3}$

Again y_{av} w.r.t. x^2 as $f(x)|_{av} = \frac{1}{(a^2-0)}\int_0^{a^2} y\, d(x^2) = \frac{b}{a^2 a}\int_0^{a^2} \sqrt{a^2 - x^2}\, dx^2 = \frac{b}{a^3}\int_0^{a^2} 2t^2\, dt = \frac{2ba^3}{3}$

10. DETERMINATION OF FUNCTION

Sometimes the area enclosed by a curve is given as a variable function and we have to find the function. The area function A_a^x satisfies the differential equation $\frac{dA_a^x}{dx} = f(x)$ with initial condition $A_a^a = 0$ i.e. derivative of the area function is the function itself. Thus we can easily find f(x) by differentiating area function.

MASTERJEE CONCEPTS

If F(x) is integral of f(x) then, $A_a^x = \int f(x)dx = [F(x) + c]$

And since, $A_a^a = 0 = F(a) + c$ $\Rightarrow c = -F(a)$.

$\therefore$ $A_a^x = F(x) - F(a)$. Finally by taking x = b we get, $A_a^b = F(b) - F(a)$

Note that this is true only if the function doesn't have any zeroes between a and b.

If the function has zero at c then area = |F(b) – F(c)| + |F(c) – F(a)|

Vaibhav Gupta (JEE 2009 AIR 54)

Illustration 30: The area from 0 to x under a certain graph is given to be $A = \sqrt{1+3x} - 1,\ x \ge 0$;

(a) Find the average rate of change of A w.r.t. x and x increases from 1 to 8.

(b) Find the instantaneous rate of change of A w.r.t. x at x = 5.

(c) Find the ordinate (height) y of the graph as a function of x.

(d) Find the average value of the ordinate (height) y, w.r.t. x as x increases from 1 to 8. **(JEE ADVANCED)**

Sol: Here by differentiating given area function we can obtain the main function.

(a) A(1) = 1, A(8) = 4 ; $\frac{A(8) - A(1)}{8-1} = \frac{3}{7}$

(b) $\left.\frac{dA}{dx}\right|_{x=5} = \left.\frac{1.\,3}{2\sqrt{1+3x}}\right|_{x=5} = \frac{3}{8}$

(c) $y = \frac{3}{2\sqrt{1+3x}}$

(d) $\frac{1}{(8-1)}\int_1^8 \frac{3}{2\sqrt{1+3x}}dx = \frac{1}{7}\int_1^8 \frac{3}{2\sqrt{1+3x}}dx = \frac{3}{7}$

Illustration 31: Let C_1 & C_2 be the graphs of the function $y = x^2$ & $y = 2x$, $0 \le x \le 1$ respectively. Let C_3 be the graphs of a function $y = f(x)$, $0 \le x \le 1$, $f(0) = 0$. For a point P on C_1, let the lines through P, parallel to the axes, meet C_2 & C_3 at Q & R respectively (see figure). If for every position of P(on C_1), the area of the shaded regions OPQ & ORP are equal, determine the function f(x). **(JEE ADVANCED)**

Sol: Similar to the above mentioned method.

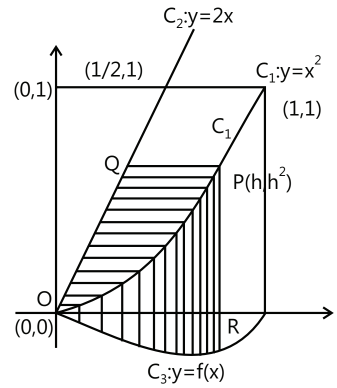


Figure 25.37

$$\int_0^{h^2}\left(\sqrt{y}-\frac{y}{2}\right)dy = \int_0^{h}(x^2 - f(x))\,dx \text{ differentiate both sides w.r.t. h}$$

$$\left(h-\frac{h^2}{2}\right)2h = h^2 - f(h)$$

$$f(h) = h^2 - \left(h-\frac{h^2}{2}\right)2h$$

$$= h^2 - h(2h - h^2) = h^2 - 2h^2 + h^3$$

$$f(h) = h^3 - h^2$$

$$f(x) = x^3 - x^2 = x^2(x - 1)$$

11. AREA ENCLOSED BY A CURVE EXPRESSED IN POLAR FORM

$r = a(1 + \cos\theta)$ (Cardioid)

$$A = \frac{1}{2}\int_0^{2\pi} r^2 d\theta = \frac{a^2}{2}\int_0^{2\pi} 4\cos^4\frac{\theta}{2}d\theta$$

Substitute $\frac{\theta}{2} = t$, $d\theta = 2dt$

$$A = a^2\int_0^{\pi} 4\cos^4 t\,dt = 8\times\frac{3\pi a^2}{16}$$

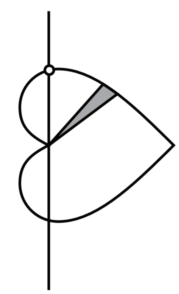


Figure 25.38

Illustration 32: Find the area enclosed by the curves $x = a\sin^3 t$ and $y = a\cos^3 t$. **(JEE MAIN)**

Sol:

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}} \quad \text{and } dx = 3a\sin^2 t.\cos t.dt$$

$$A = 4\int_0^{a} y\,dx \;;\; A = 4a^2\int_0^{\pi/2} 3\cos^3 t\sin^2 t\cos t\,dt$$

$$A = 12a^2\int_0^{\pi/2}\sin^2 t\cos^4 t\,dt = (12a^2)\,.\,\frac{1.3.1}{6.4.2}\,.\,\frac{\pi}{2} = \frac{12a^2\pi}{32} = \frac{3\pi a^2}{8}$$

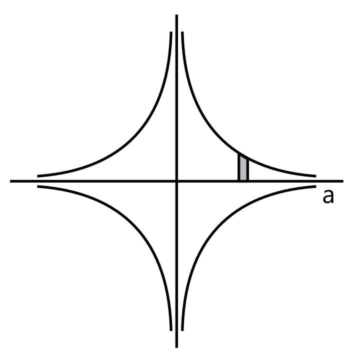


Figure 25.39

Linear Programming

1. INTRODUCTION

Linear Programming was developed during World War II, when a system with which to maximize the efficiency of resources was of utmost importance.

2. LINEAR PROGRAMMING

Linear programming may be defined as the problem of maximising or minimising a linear function subject to linear constraints. The constraints may be equalities or inequalities. Here is an example.

Find numbers x_1 and x_2 that maximize the sum $x_1 + x_2$ subject to the constraints $x_1 \geq 0$, $x_2 \geq 0$, and

$$x_1 + 2x_2 \leq 4$$

$$4x_1 + 2x_2 \leq 12$$

$$-x_1 + x_2 \leq 1$$

Here we have two unknowns and five inequalities (constraints). Notice that these constraints are all linear functions of the variables. The first two constraints, $x_1 \geq 0$ and $x_2 \geq 0$, are special. These are called no negativity constraints and are often found in linear programming problems. The other constraints are called the main constraints. The function to be maximised (or minimized) is called the objective function. In the above example the objective function is $x_1 + x_2$.

3. GRAPHICAL METHOD

As we have only two variables, we can solve this problem by plotting the constraints with x_1 and x_2 as axes. The intersection region of these inequalities is called feasible region for the objective function. This is the region which satisfies all the constraints. Now from this feasible region we have to select point(s) such that objective function is maximized or minimized.

Theorem 1: Let R be the feasible region (convex polygon) for a linear programming problem and let Z=ax + by be the objective function. When Z has an optimal value (maximum or minimum), where the variables x and y are subject to constraints described by linear inequalities, this optimal value must occur at a corner point (vertex) of the feasible region.

Theorem 2: Let R be the feasible region for a linear programming problem and let Z=ax + by be the objective function. If R is bounded, then the objective function Z has both a maximum and a minimum value on R and each of these occurs at corner point (vertex) of R.

Remark: If R is unbounded, then a maximum or a minimum value of the objective function may not exist. However, if it exists it must occur at a corner point of R. (By Theorem 1).

So for the above example

Corner point (x_1, x_2)	Z (= $x_1 + x_2$) value
0,1	1
3,0	3
8/3,2/3	10/3
2/3,5/3	7/3

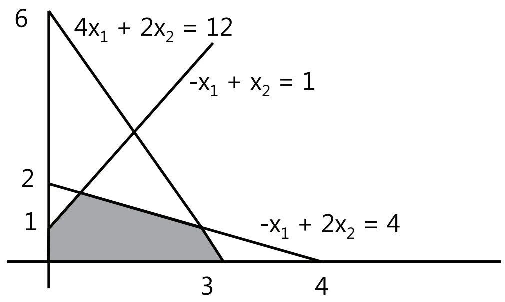


Figure 25.40

Hence (8/3, 2/3) is the optimal solution.

Note that z has also minimum value in the feasible region at (0, 1).

This method of solving is generally called as corner point method. Note that a function can have more than one optimal points.

4. MODELS

There are few important linear programming models which are more frequently used and some of them we encounter in our daily lives.

(a) Manufacturing/Assignment problems: In these problems, we determine the number of units of different products which should be produced and sold by a firm when each product requires a fixed manpower, machine hours, labour hours per unit of product, warehouse space per unit of the output. In order to make maximum profit.

Example: There are I persons available for J jobs. The value of person i working 1 day at job j is a_{ij}, for i = 1,......,I, and j = 1,.....,J. The problem is to choose an assignment of persons to jobs to maximize the total value.

An assignment is a choice of numbers, x_{ij}, for i = 1,......,I, and j=1,......,J, where x_{ij} represents the proportion of person i's time that is to be spent on job j. Thus,

$$\sum_{j=1}^{J} x_{ij} \le 1 \quad \text{For } i = 1,\ldots\ldots, I \qquad \ldots \text{(i)}$$

$$\sum_{i=1}^{I} x_{ij} \le 1 \quad \text{For } j=1,\ldots\ldots, J \qquad \ldots \text{(ii)}$$

And $x_{ij} \ge 0$ for i = 1,......,I, and j=1,......,J ... (iii)

Equation (i) reflects the fact that a person cannot spend more than 100% of his time working, (ii) means that only one person is allowed on a job at a time, and (iii) says that no one can work a negative amount of time on any job, Subject to (i), (ii) and (iii) , we wish to maximize the total value of $\sum_{i=1}^{I}\sum_{j=1}^{J} a_{ij}x_{ij}$

(b) Diet problems: In these problems, we determine the amount of different kinds of nutrients which should be included in a diet so as to minimise the cost of the desired diet such that it contains a certain minimum amount of each nutrients.

Example: There are m different types of food, $F_1,\ldots\ldots\ldots,F_m$, that supply varying quantities of the n nutrients , $N_1,\ldots\ldots\ldots,N_n$, that are essential to good health. Let c_j be the minimum daily requirement of nutrient, N_j contained in one unit of food F_i. The problem is to supply the required nutrients at minimum cost.

Let y_i be the number of units of food F_i to be purchased per day. The cost per day of such a diet is

$$b_1y_1 + b_2y_2 + \ldots\ldots\ldots + b_my_m \qquad \ldots \text{(i)}$$

The amount of nutrient N_j contained in this diet is

$$a_{1j}y_1 + a_{2j}y_2 + \ldots\ldots\ldots + a_{mj}y_m$$

For j = 1,........, n. We do not consider such a diet unless all the minimum daily requirements are met, that is, unless

$$a_{1j}y_1 + a_{2j}y_2 + \ldots\ldots\ldots + a_{mj}y_m \ge c_j \qquad \text{For } j = 1,\ldots\ldots\ldots, n \qquad \ldots \text{(ii)}$$

Of course, we cannot purchase a negative amount of food, so we automatically have the constraints

$$y_1 \ge 0,\ y_2 \ge 0, \ldots\ldots\ldots\ldots\ldots\ldots, y_m \ge 0 \qquad \ldots \text{(iii)}$$

Our problem is: minimize (i) subject to (ii) and (iii). This is exactly the standard minimum problem.

(c) Transportation problems: In these problems, we determine a transportation schedule in order to find the cheapest way of transporting a product from plants/factories situated at different locations to different markets.

Example: There are I ports, or production plants, $P_1,.........P_I$, that supply a certain commodity, and there are J markets, $M_1,.........M_J$, to which this commodity must be shipped. Port P_i possesses an amount s_i of the commodity (i=1,2,......I), and market M_j must receive the amount r_j of the commodity (j = 1,........J). Let b_{ij} be the cost of transporting one unit of the commodity from port P_i to market M_j. The problem is to meet the market requirements at minimum transportation cost is

$$\sum_{i=1}^{I}\sum_{j=1}^{J} y_{ij}b_{ij} \qquad \text{... (i)}$$

The amount sent from port P_i is $\sum_{j=1}^{J} y_{ij} \le y_{ij}$ and since the amount available at port P_i is s_i, we must have

$$\sum_{j=1}^{J} y_{ij} \le s_i \quad \text{for } i = 1,......,I \qquad \text{... (ii)}$$

The amount sent to market M_j is $\sum_{i=1}^{I} y_{ij}$, and since the amount required there is r_j, we must have

$$\sum_{i=1}^{I} y_{ij} \le r_j \quad \text{for } j = 1,......,I \qquad \text{... (iii)}$$

It is assumed that we cannot send a negative amount from P_I to M_j, we have

$$y_{ij} \ge 0 \quad \text{for } I = 1,........I \text{ and } j = 1,........J. \qquad \text{... (iv)}$$

Our problem is minimize (i) subject to (ii), (iii) and (iv).

FORMULAE SHEET

(a) Area bounded by a curve with x – axis: Area $= \int_a^b y\,dx = \int_a^b f(x)\,dx$

(b) Area bounded by a curve with y – axis: Area $= \int_c^d x\,dy = \int_c^d f(y)\,dy$

(c) Area of a curve in parametric form: Area $= \int_a^b y\,dx = \int_{t_2}^{t_1} g(t)f'(t)\,dt$

(d) Positive and Negative Area: $A = \left|\int_a^c f(x)\,dx\right| + \left|\int_c^b f(x)\,dx\right|$;

(e) Area between two curves:

(i) Area enclosed between two curves intersecting at two different points.

Area $= \int_a^b (y_1 - y_2)\,dx = \int_a^b [f_1(x) - f_2(x)]\,dx$

(ii) Area enclosed between two curves intersecting at one point and the x – axis.

Area $= \int_a^{\alpha} f_1(x)\,dx + \int_{\alpha}^{b} f_2(x)\,dx$

(iii) Area bounded by two intersecting curves and lines parallel to y – axis.

Area $= \int_a^c (f(x) - g(x))\,dx + \int_c^b (g(x) - f(x))\,dx$

(a) Standard Areas:

(i) Area bounded by two parabolas $y^2 = 4ax$ and $x^2 = 4by$; $a > 0$, $b > 0$: Area $= \frac{16ab}{3}$

(ii) Area bounded by Parabola $y^2 = 4ax$ and Line $y = mx$: Area $= \frac{8a^2}{3m^3}$

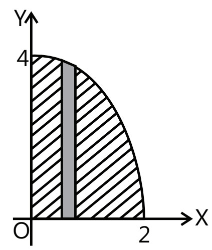
(iii) Area of an Ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$: Area $= \pi ab$

Solved Examples

JEE Main/Boards

Example 1: Find area bounded by $y = 4 - x^2$, x-axis and the lines $x = 0$ and $x = 2$.

Sol: By using the formula of Area Bounded by the x – axis, we can obtain

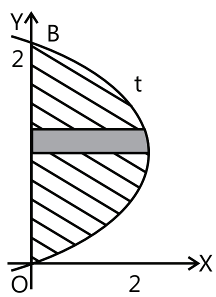


Required Area.

$$= \int_0^2 y\,dx = \int_0^2 (4 - x^2)\,dx$$

$$= \left(4x - \frac{x^3}{3}\right)_0^2 = 8 - \frac{8}{3} = \frac{16}{3} \text{ sq. units}$$

Example 2: Find the area bounded by the curve $y^2 = 2y - x$ and the y-axis.



Sol: Here given equation is the equation of parabola with vertex (1, 1) and curve passes through the origin.

Curve is $y^2 - 2y = -x$ or $(y - 1)^2 = -(x - 1)$

It is a parabola with

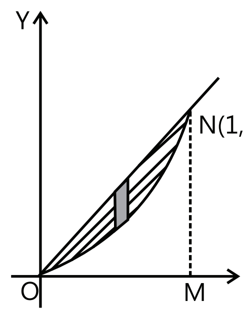
Vertex at (1, 1) and the curve passes through the origin. At B, $x = 0$ and $y = 2$

Area

$$= \int_0^2 x\,dy = \int_0^2 (2y - y^2)\,dy = \left(y^2 - \frac{y^3}{3}\right)_0^2 = \frac{4}{3} \text{ sq. units}$$

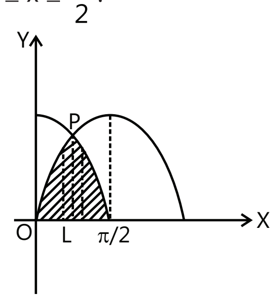
Example 3: Find the area of the region $\{(x, y): x^2 \le y \le x\}$

Sol: Consider the function $y = x^2$ and $y = x$ Solving them, we get $x = 0$, $y = 0$ and $x = 1$, $y = 1$; $x^2 \le y \Rightarrow$ area is above the curve $y = x^2$ $y \le x \Rightarrow$ area is below the line $y = x$



$$\text{Area} = \int_0^1 (x - x^2)\,dx = \left(\frac{x^2}{2} - \frac{x^3}{3}\right)_0^1 = \frac{1}{6} \text{ sq. units}$$

Example 4: Find the area of the region enclosed by $y = \sin x$, $y = \cos x$ and x-axis, $0 \le x \le \frac{\pi}{2}$.



Sol: Find point of intersection is P. Therefore after obtaining the co-ordinates of P and then integrating with appropriate limits, we can obtain required Area.

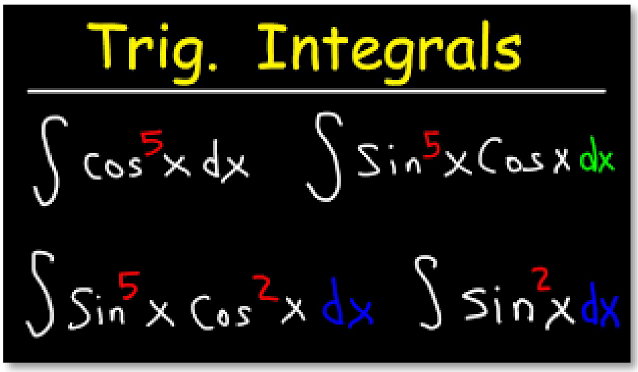
At point of intersection P, $x = \frac{\pi}{4}$ as ordinates of $y = \sin x$ and ; $y = \cos x$ are equal

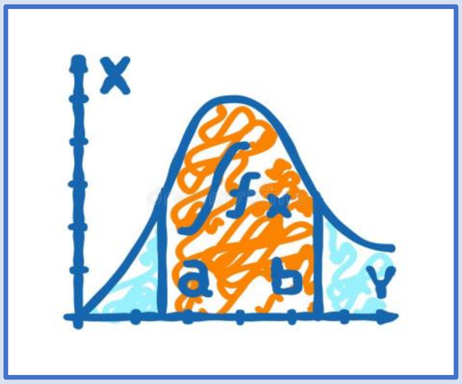
Hence, P is $\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right)$ Required area

$$= \int_0^{\pi/4} \sin x\,dx + \int_{\pi/4}^{\pi/2} \cos x\,dx = \left(-\cos x\right)_0^{\pi/4} + \left(\sin x\right)_{\pi/4}^{\pi/2}$$

$$= \left(-\frac{1}{\sqrt{2}} + 1\right) + \left(1 - \frac{1}{\sqrt{2}}\right) = 2 - \sqrt{2} \text{ sq. units}$$

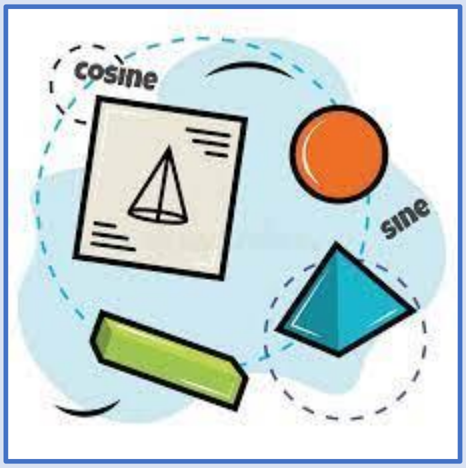
Lecture Fourteen





Behavioral objectives

1. Recall and apply standard integral formulas for trigonometric functions.
2. Prove integration identities for sine, cosine, secant, and cosecant functions.
3. Evaluate integrals involving trigonometric identities using substitution techniques.
4. Solve problems using antiderivatives of basic trigonometric forms.



INTEGRATION OF TRIGONOMETRIC FUNCTIONS

For this part of the topic, we will acquaint ourselves with the first six formulas for the *antiderivatives of trigonometric functions*. These formulas in general, will just come from the formulas of the derivatives of the six trigonometric functions. We will state first the formulas in terms of theorems, and we will prove them one by one.

Theorem 6: $\int \sin x dx = -\cos x + C$

Proof: Since the antiderivative of *sin x* is *–cos x + C*, we must show that the derivative of *–cos x + C* is indeed, *sin x*

$\Rightarrow D_x(-\cos x + C) = D_x(-\cos x) + D_x(C)$ Recall: $D_x(\cos x) = -\sin x$

$= -D_x(\cos x) + 0 = -(-\sin x)$

$\therefore \boxed{D_x(-\cos x + C) = \sin x}$ ▪

Note: The integral of the form sin x is NEGATIVE *cos x + C*, unlike its derivative which has a positive sign, that is $D_x(\sin x) = \cos x$. We must not confuse ourselves with the signs of the answers for the derivative and integral of *sin x*.

Theorem 7: $\int \cos x dx = \sin x + C$

Proof: We must show that the derivative of sin x + C is cos x.

$\Rightarrow D_x(\sin x + C) = D_x(\sin x) + D_x(C)$ Recall: $D_x(\sin x) = \cos x$

$\therefore \boxed{D_x(\sin x + C) = \cos x}$ ▪

Note: In this case the integral of the form cos x is POSITIVE *sin x + C*, unlike its derivative having a negative sign, that is $D_x(\cos x) = -\sin x$. We must not confuse ourselves with the signs of the answers for the derivative and integral of *cos x*.

Theorem 8: $\int \sec^2 x dx = \tan x + C$

Proof: The proof of this theorem is straightforward, since the derivative of *tan x + C* is just $sec^2 x$. ▪

Theorem 9: $\int \csc^2 x dx = -\cot x + C$

Proof: To show that the derivative of –*cot x* + *C* is *csc² x*,

$\Rightarrow D_x(-\cot x + C) = D_x(-\cot x) + D_x(C)$ Recall: $D_x(\cot x) = -\csc^2 x$

$= -D_x(\cot x) + 0 = -(-\csc^2 x)$

$\therefore \boxed{D_x(-\cot x + C) = \csc^2 x}$ ▪

Theorem 10: $\int \tan x \sec x dx = \sec x + C$

Proof: The proof of this theorem is straightforward, since the derivative of *sec x* + *C* is just *tan x sec x*. ▪

Theorem 11: $\int \cot x \csc x dx = -\csc x + C$

Proof: To show that the derivative of –*csc x* + *C* is *cot x csc x*,

$\Rightarrow D_x(-\csc x + C) = D_x(-\csc x) + D_x(C)$ Recall: $D_x(\csc x) = -\cot x \csc x$

$= -D_x(\csc x) + 0 = -(-\cot x \csc x)$

$\therefore \boxed{D_x(-\csc x + C) = \cot x \csc x}$ ▪

Let's now summarize the six formulas that we had proven:

1. $\int \sin x dx = -\cos x + C$	4. $\int \csc^2 x dx = -\cot x + C$
2. $\int \cos x dx = \sin x + C$	5. $\int \tan x \sec x dx = \sec x + C$
3. $\int \sec^2 x dx = \tan x + C$	6. $\int \cot x \csc x dx = -\csc x + C$

Note that these six formulas are just preliminary formulas for the integrals of trigonometric functions. These are not yet complete and we will still encounter other formulas in our succeeding discussions.

In the examples that will follow, we are to devise a way to use these formulas by possibly using some trigonometric identities to execute the solution correctly. We can also use the preliminary theorems in the previous topic if needed.

Recall: The Eight Fundamental Identities

Reciprocal Identities:	$\frac{1}{\sin x} = \csc x$	;	$\frac{1}{\csc x} = \sin x$	(1)
	$\frac{1}{\cos x} = \sec x$	;	$\frac{1}{\sec x} = \cos x$	(2)
	$\frac{1}{\tan x} = \cot x$	;	$\frac{1}{\cot x} = \tan x$	(3)
Quotient Identities:	$\frac{\sin x}{\cos x} = \tan x$	(4)	$\frac{\cos x}{\sin x} = \cot x$	(5)
Pythagorean Identities:	$1 - \sin^2 x = \cos^2 x$	;	$1 - \cos^2 x = \sin^2 x$	(6)
	$1 + \tan^2 x = \sec^2 x$	;	$\sec^2 x - 1 = \tan^2 x$	(7)
	$1 + \cot^2 x = \csc^2 x$	;	$\csc^2 x - 1 = \cot^2 x$	(8)

Example 1: $\int \left(3\sec x \tan x - 5\csc^2 x\right)dx$

We can first use some preliminary theorems first before integrating the trigonometric functions involved. Distributing the integration symbol and taking out constant multiples first, we have:

$$\Rightarrow \int \left(3\sec x \tan x - 5\csc^2 x\right)dx = \int 3\sec x \tan x dx - \int 5\csc^2 x dx$$

$$= 3\int \sec x \tan x dx - 5\int \csc^2 x dx$$

Noting that $\int \sec x \tan x dx = \sec x + C$ and $\int \csc^2 x dx = -\cot x + C$ and just annexing +*C* at the last term:

$$\Rightarrow \int \left(3\sec x \tan x - 5\csc^2 x\right)dx = 3(\sec x) - 5(-\cot x) + C$$

$$\therefore \int \left(3\sec x \tan x - 5\csc^2 x\right)dx = \boxed{3\sec x + 5\cot x + C}$$

WARNING: DO NOT FORGET TO ANNEX THE +*C* TERM IN YOUR FINAL ANSWER for all indefinite integrals. Recall that the additional arbitrary constant is a very important concept involving the antiderivatives of any functions.

Example 2: $\int \frac{2\cot x - 3\sin^2 x}{\sin x}dx$

Since the form of the expression that we will integrate (integrand) does not satisfy yet any of the formulas that we have so far, we simplify the given first by dividing each term of the numerator by sin x, and we have:

$$\Rightarrow \int \frac{2\cot x - 3\sin^2 x}{\sin x}dx = \int\left(\frac{2\cot x}{\sin x} - \frac{3\sin^2 x}{\sin x}\right)dx = \int\left(\frac{2\cot x}{\sin x} - 3\sin x\right)dx$$

$$= \int \frac{2\cot x}{\sin x}dx - \int 3\sin x dx = 2\int \frac{\cot x}{\sin x}dx - 3\int \sin x dx$$

Note that the second term is now possible to integrate since we now have a ready-made formula for it. For the first term, we must think of a way to use our available formulas, by using some trigonometric identities. One way is by isolating sin x and use a reciprocal identity, thus:

$$\Rightarrow \int \frac{2\cot x - 3\sin^2 x}{\sin x}dx = 2\int \cot x\left(\frac{1}{\sin x}\right)dx - 3\int \sin x dx \qquad ; \qquad \frac{1}{\sin x} = \csc x$$

$$= 2\int \cot x \csc dx - 3\int \sin x dx = 2(-\csc x) - 3(-\cos x) + C$$

$$\therefore \int \frac{2\cot x - 3\sin^2 x}{\sin x}dx = \boxed{-2\csc x + 3\cos x + C}$$

Example 3: $\int\left(\tan^2 x + \cot^2 x + 4\right)dx$

For us to use the formulas that we know, we can transform the first two terms in terms of their respective Pythagorean identities first:

$$\Rightarrow \int\left(\tan^2 x + \cot^2 x + 4\right)dx = \int\left[\left(\sec^2 x - 1\right) + \left(\csc^2 x - 1\right) + 4\right]dx$$

$$= \int\left(\sec^2 x + \csc^2 x + 2\right)dx = \int \sec^2 x dx + \int \csc^2 x dx + 2\int dx$$

$$\therefore \int\left(\tan^2 x + \cot^2 x + 4\right)dx = \boxed{\tan x - \cot x + 2x + C}$$

These examples will serve as model examples for now, wherein we will still encounter or use these formulas as we proceed with the topics.

EXERCISE 1.3: Evaluate the following indefinite integrals.

1. $\int \frac{\sin x}{\cos^2 x} dx$ (Ans: $\sec x + C$)

2. $\int \left(4\csc x \cot x + 2\sec^2 x\right) dx$

3. $\int \left(3\csc^2 t - 5\sec t \tan t\right) dt$

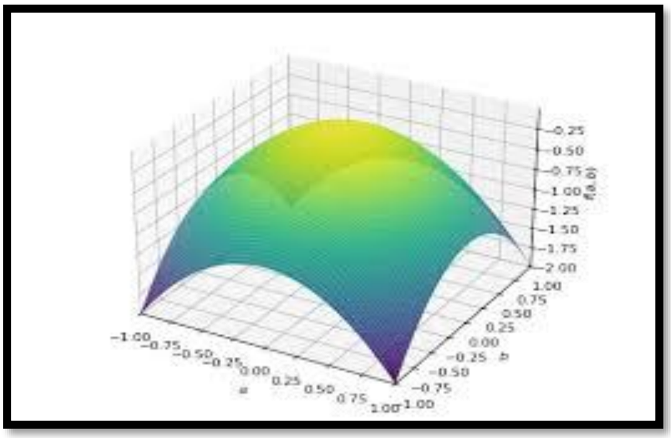
4. $\int \frac{3\tan\theta - 4\cos^2\theta}{\cos\theta} d\theta$

5. $\int \frac{dx}{1 - \sin x}$ (Ans: $\tan x + \sec x + C$)

   Hint: multiply numerator and denominator by 1 + *sin x*

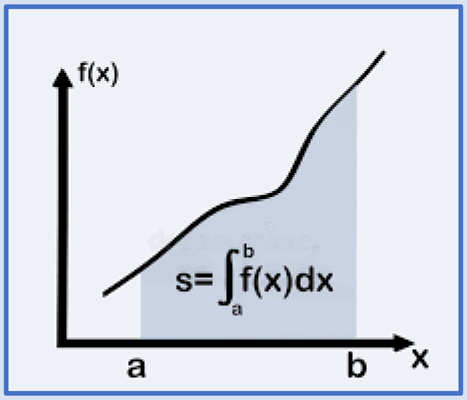
6. $\int (2\tan x - \cot x)^2 dx$ (Ans: $4\tan x - \cot x - 9x + C$)

Lecture Fifteen



Behavioral objectives

1. Compute definite integrals using the fundamental theorem of calculus.
2. Evaluate definite integrals with limits to find exact area values.
3. Distinguish between positive and negative area under a curve based on the graph.
4. Apply integration to find areas between curves and solve applied mathematical problems.



The Definite Integral and Applications

The Definite Integral gives a number. We use integration rules to find the integral, omit the +C (even if we put it it will cancel out so we don't bother) and evaluate the integral at the top and bottom limits and take the difference.

$$\int_0^3 2x + 5\,dx = x^2 + 5x\Big|_0^3 = \left(3^2 + 5\cdot 3\right) - \left(0^2 + 5\cdot 0\right) = 24 - 0 = 24$$

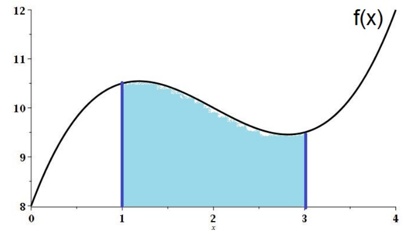
Application: Computing Areas Using Integrals

If $f(x) > 0$ and the graph is above the x-axis, the integral gives the ares under the curve and above the x-axis. The graph on the right is of

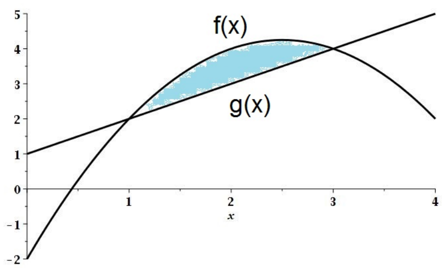
$$f(x) = 0.5x^3 - 3x^2 + 5x + 8$$

so the area shown is given by the definite integral

$$\int_1^3 0.5x^3 - 3x^2 + 5x + 8\,dx.$$



If the function $f(x) < 0$, the integral is negative of the area.



If the area is between two graphs, we integrate "top curve minus bottom curve". On the right are the graphs of $f(x) = -x^2+5x-2$ and $g(x) = x+1$. The area between them is given by the integral

$$\int_1^3 \left(-x^2 + 5x - 2\right) - (x+1)\,dx$$

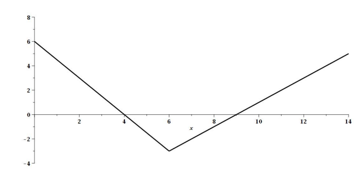
If you cannot see clearly the x-values for the intersection points from the graph, you can solve for them by setting the two equations equal to each other. In this case

$$-x^2 + 5x - 2 = x + 1.$$

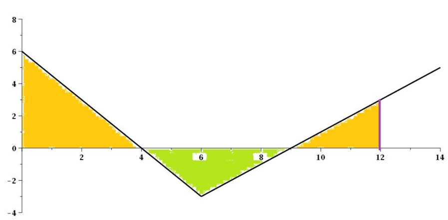
Application: Using Areas to Compute Integrals

To find $\int_0^{12} f(x)\,dx$ for the function whose graph is given, we could work out the equations of the two lines making up $f(x)$ using the points on the graph. If we did, the answer would look like this:

$$\int_0^{12} f(x)\,dx = \int_0^6 -\frac{3}{2}x + 6\,dx + \int_6^{12} \frac{5}{6}x - 5\,dx$$



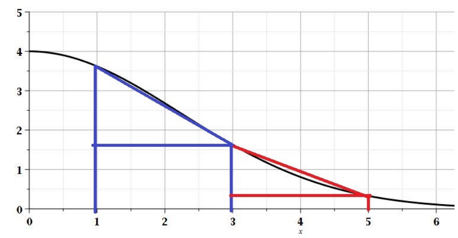
The integral is split because the formula changes at $x = 6$. You can see why this method is long and not practical.



It is better to interpret the integral as area. Since areas under the x- axis give negative integrals, we add the blue areas and subtract the green area (all triangles) to calculate the integral:

$$\int_0^{12} f(x)\,dx = \frac{4\times 6}{2} - \frac{5\times 2.5}{2} + \frac{3\times 2.5}{2}$$

If we are given the graph of the function instead of its formula, we can approximate the integral by approximating the area. To approximate area under a curve, draw some rectangles and triangles very close to the actual curve and compute those areas.



Application: The Definite Integral of a Rate of Change Gives the Change

This is the most important application of integration for us. In math notation it says

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

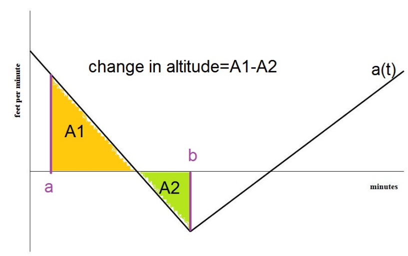
We integrate rate of change to get total change.

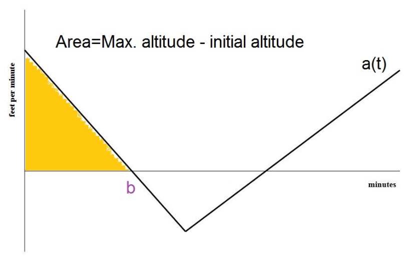
Change of Altitude from Rate of Ascent

For example, if $a(t) = A'(t)$ is rate of ascent of a balloon - so it is the rate of change of its height $A(t)$

$$\int_a^b a(t)\,dt = \int_a^b A'(t)\,dt = A(b) - A(a) \quad \text{is the change in altitude.}$$

Areas under the t axis will make a negative contribution to the integral.





At the point where $a(t)$ switches from possessive to negative, $t = b$ in picture, the altitude makes a maximum value.

$$\int_0^b a(t)\,dt = A(b) - A(0) = \text{Maximum Altitude} - \text{Initial Altitude}$$

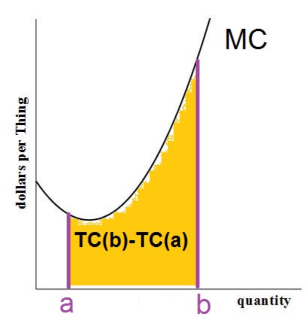
Change in Cost and Revenue from Marginal Cost and Marginal Revenue

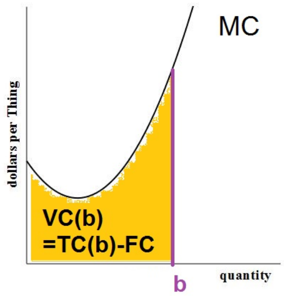
If we integrate Marginal Cost, we get the change in Total Cost:

$$\int_a^b MC(q)\,dq = TC(b) - TC(a).$$

We also get the change in Variable Cost because MC is also the derivative of Variable Cost:

$$\int_a^b MC(q)\,dq = VC(b) - VC(a)$$





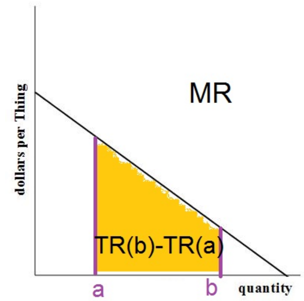
In particular, since $VC(0) = 0$

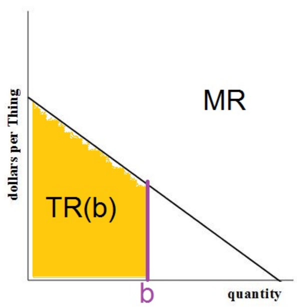
$$\int_0^b MC(q)\,dq = VC(b) - VC(0) = VC(b)$$

So we can use an integral of MC to compute VC at any quantity.

If we integrate Marginal Revenue, we get the difference in Total Revenue

$$\int_a^b MC(q)\,dq = TR(b) - TR(a)$$





In particular, since $TR(0) = 0$

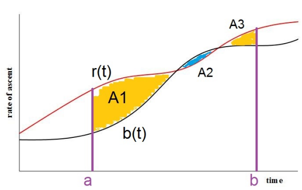
$$\int_0^b MR(q)\,dq = TR(b) - TR(0) = TR(b)$$

So we can use an integral of MR to compute TR at any quantity.

Application: Working with Two Rates

The integral of a difference can be interpreted as area between two functions. Since $(f - g)' = f' - g'$, integrating a difference of two rates gives the change of their difference.

Two Rates of Ascent

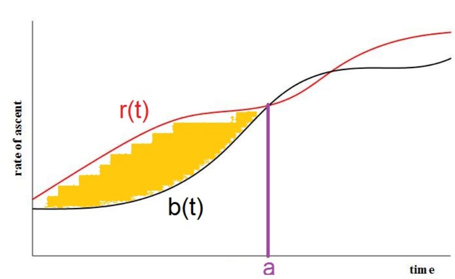


On the left are graphs of rates of ascent $r(t)$ and $b(t)$for two balloons, Red and Black. Initially they are at the same altitude. When $r(t) < b(t)$, the Red balloon will rise faster increasing the distance between them. When $r(t) < b(t)$, the Black balloon will start to catch up decreasing the distance between them.

$$\int_a^b r(t) - b(t)\,dt = \text{change in the distance between the balloons} = A1 - A2 + A3$$

We know maximum distance is attained when we switch form $r(t) > b(t)$ to $r(t) < b(t)$. Say it happens at $t = a$. If they initially start at the same height, then,

$$\int_0^a r(t) - b(t) = \text{Maximum Distance between them} = \text{area}.$$

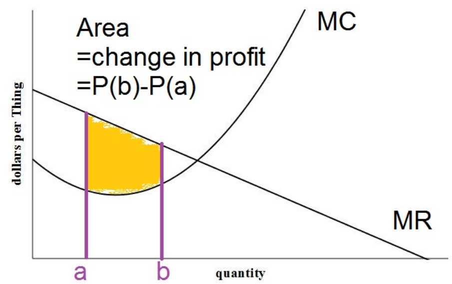


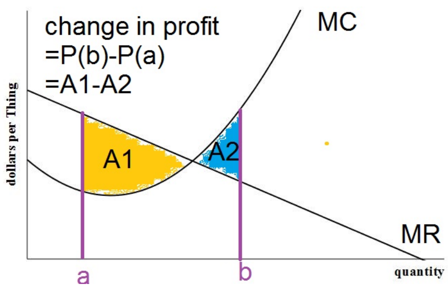
Marginal Revenue and Marginal Cost Together

Since $P(q) = TR(q) - TC(q)$ we have $P'(q) = MR(q) - MP(q)$ so

$$\int_a^b MR(q) - MC(q)\,dq = P(b) - P(a) = \text{change in Profit},$$

When we have the graphs of MR and MC we can interpret this as the area between two functions.



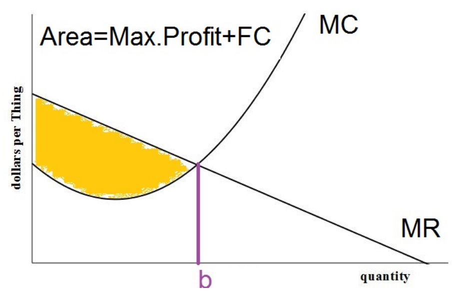


When $MR < MC$, profits start decreasing. So in the integral, area between the two will make a negative contribution to the change in Profit.

We know **maximum profit** is attained when we switch form $MR > MC$ to $MR < MC$. This is exactly the place where we switch from $P' > 0$ to $P' < 0$.(Why?) Say it happens at $q = b$. Then,

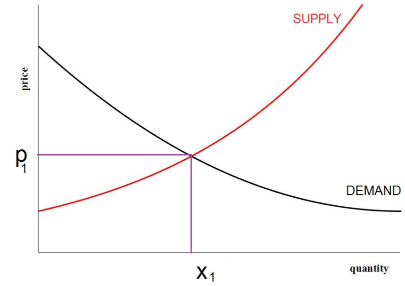
$$\int_0^b MR(q) - MC(q) = P(b) - P(0)$$

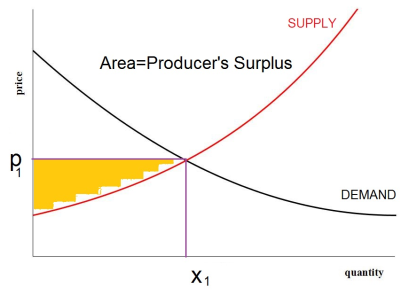
$$= P(b) - (-FC) = \text{Maximum Profit} + FC.$$



Application: Producer's and Consumer's Surplus

Recall the basic facts about Supply and Demand: The Supply function is increasing. When prices go up, so does the quantity supplied. The Demand function is decreasing. When prices go up, the quantity decreases as fewer people are willing to buy. Also, The quantity (q or x) is always on the x-axis, and price p is on the y-axis. Their intersection point (x_1, p_1), whcih you can find by setting the demand and supply equations equal to each other, is called the **equilibrium point**.





The Producer's Surplus is given by

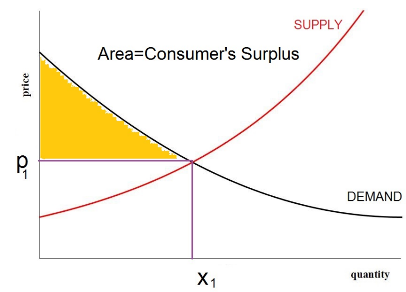
$$p_1 x_1 - \int_0^{x_1} \text{"Supply Function"}\, dx.$$

It is the shaded area on the left.

The Consumer's Surplus is given by

$$\int_0^{x_1} \text{"Demand Function"}\, dx - p_1 x_1$$

It is the shaded area on the right.



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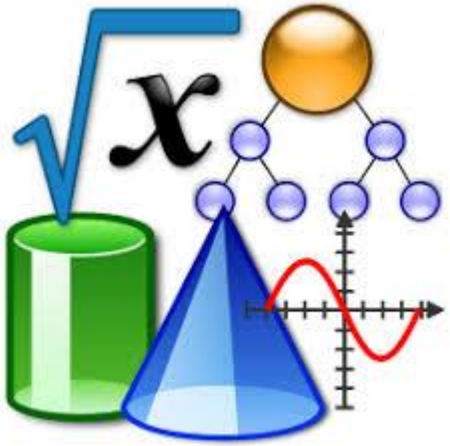

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